Entangled graphs: Bipartite entanglement in multi-qubit systems

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I. INTRODUCTION

The entanglement is a key ingredient of quantum mechanics [1,2]. In the last decade it has been identified as a key resource for quantum information processing. In particular, quantum computation [3,4], quantum teleportation [5], quantum dense coding [6], certain types of quantum key distributions [7] and quantum secret sharing protocols [8], are based on the existence of entangled states.

The nature of quantum entanglement between two qubits is well understood now. In particular, the necessary and sufficient condition for inseparability of two-qubit systems has been derived by Peres [9] and Horodecki et al. [10]. Reliable measures of bi-partite entanglement have been introduced and well analyzed (see for instance Refs. [11,12]). On the other hand it is a very difficult task to generalize the analysis of entanglement from two to multi-partite systems. The multi-partite entanglement is a complex phenomenon. One of the reasons is that quantum entanglement cannot be shared freely among many particles. For instance, having four qubits, we are able to prepare a state with two e-bits (two Bell pairs, as an example), but not more. This means that the structure of quantum mechanics imposes strict bounds on bi-partite entanglement in multi-partite systems. This issue has been first addressed by Wootters et al. [13,14] who have derived important bounds on shared bi-partite entanglement in multi-qubit systems. In fact, one can solve a variational problem to answer a question: What is a pure multi-partite state with specific constraints on bi-partite entanglement? O'Connors and Wootters [14] have studied what is the state of a multi-qubit ring with maximal possible entanglement between neighboring qubits. Another version of the same problem has been analyzed by Koashi et al. [15] who have derived an explicit expression for the multi-qubit completely symmetric state (entangled web) in which all possible pairs of qubits are maximally entangled.

Following these ideas we analyze in the present paper a new object, the so-called *entangled graph*. In the graph, each qubit is represented as a vertex and an edge between two vertices denotes entanglement between these two particles (specifically, the corresponding two-qubit density operator is inseparable). The central issue of the paper is to show that any entangled graph with N vertices and k edges can be associated with a *pure* multi-qubit state. We prove this result constructively, by showing the explicit expression of corresponding pure states. We show that any entangled graph of N qubits can be represented by a pure state from a subspace of the whole 2^N-dimensional Hilbert space of N qubits. The dimension of this subspace is at most quadratic in number of qubits.

In some sense entangled graphs are objects similar to those studied recently by Dür [16]. He investigated, how to prepare multi-qubit states with specific pairs of qubits being entangled. However, Dür did not take into account the condition of separability between the rest of pairs of qubits in the system. Certainly, our approach is much more complex, with many more constraints since in the entangled graph we have to fulfill all conditions for inseparability as well as separability between specific qubits.

II. ENTANGLED GRAPHS

Let us consider entangled graphs associated with pure N-qubit states. These graphs consist of N vertices. Let the parameter k denote the number of edges in the graph, with the condition

$$0 \leq k \leq \frac{N(N-1)}{2}. \tag{2.1}$$

Then let us define a set S with k members. These will be pairs of qubits between which we expect entanglement; thus for every $i<j$

$$\begin{aligned} \{i,j\} \in S \quad &\Longleftrightarrow \quad C(i,j) > 0 \\ \{i,j\} \notin S \quad &\Longleftrightarrow \quad C(i,j) = 0. \end{aligned} \tag{2.2}$$

In what follows we propose a general algorithm how to construct a pure state for an arbitrary graph. Let us consider a pure state of N ($N>4$) qubits described by the vector

$$|\Psi\rangle = \alpha|0..0\rangle + \beta|1..1\rangle + \sum_{\{i,j\}\in S} \frac{\gamma}{\sqrt{k}}|1\rangle_i|1\rangle_j|0...0\rangle_{\overline{ij}} \tag{2.3}$$

with the normalization condition $|\alpha|^2+|\beta|^2+|\gamma|^2=1$.

One can check that there are many states which fulfill the conditions for concurrences. In particular, let us assume the state (2.3) with

$$\alpha = \frac{k}{\sqrt{k^2+2k+4}}\,; \beta = \frac{2\alpha}{k}\,; \gamma = \alpha\sqrt{\frac{2}{k}}\,. \tag{2.4}$$

This state indeed corresponds to the desired graph. This proves that one can associate with an arbitrary entangled graph a pure state. Moreover, by construction we have proved that in general this state is a superposition of at most N^2 vectors from the 2^N-dimensional Hilbert space of N qubits.

III. CONCLUSION

We have proposed a method for characterization of two-particle entanglement in multi-qubit systems: We have introduced a new concept of *entangled graphs*: every qubit is represented by a vertex while entanglement between two qubits is represented as an edge between relevant vertices. We have shown that for every possible graph with non-weighted edges there exists a pure state, which represents the graph. Moreover, such state can be constructed as a superposition of small number of states from a subspace of the Hilbert space. The dimension of this subspace grows linearly with the number of entangled pairs (thus, in the worst case, quadratically with the number of particles).

Introducing weight to edges in the graphs would allow us to optimize entanglement for certain graphs. It could even lead to entanglement engineering, when for every pair we could specify the relative strength of entanglement and find the optimal state with maximal concurrencies, as it was made for specific cases in Refs. [14,15].

States with defined bipartite entanglement properties are also of a possible practical use. In communications protocols, like quantum secret sharing [8] or quantum oblivious transfer [17] one needs many-particle states with specific bipartite entanglement properties. The one-way quantum computer, suggested by Briegel et al. [18] performs quantum computation only via projective measurement. For this purpose, one needs to prepare a "substrate", a complex cluster state of many qubits. These states can also be associated with entangled graphs. Therefore deep understanding of possible entangled graphs can help us to understand structure of quantum correlation and the corresponding bounds on quantum communications and quantum information processing.

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[1] E. Schrödinger, Naturwissenschaften **23**, 807, (1935); *ibid.* **23**, 823 (1935); *ibid.* **23**, 844 (1935).
[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. A **47**, 777 (1935); J.S.Bell, Physics **1**, 195 (1964); A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1993).
[3] J. Gruska, *Quantum Computing* (McGraw-Hill,1999); J. Preskill, *Quantum Theory Information and Computation* (`www.theory.caltech.edu/people/preskill`).
[4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[5] C. H. Bennett, *et al.*, Phys. Rev. Lett. **70**, 1895 (1993).
[6] C. H. Bennett and S. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[7] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[8] M. Hillery, V. Bužek, and A. Berthiaume Phys. Rev. A **59**, 1829 (1999); R. Cleve, D. Gottesman, H. Lo, Phys. Rev. Lett. **83**, 1874 (1999).
[9] A. Peres, Phys. Rev. Lett. **77**, 4524 (1996).
[10] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[11] V. Vedral, M.B. Plenio, M.A. Rippin, and P.L.Knight, Phys. Rev. Lett. **78**, 2275 (1997).
[12] G. Alber, T. Beth, M. Horodecki, P. horodecki, R. Horodecki, M Rötteler, H. Winfurter, R.F. Werner, A. Zeilinger, Quantum information, Berlin (2001)
[13] V. Coffman, J. Kundu, W. K. Wootters, Phys. Rev. A **61**, 052306 (2000)
[14] K.M. O'Connor and W.K. Wootters, Phys. Rev. A **63**, 052302 (2001).
[15] M. Koashi, V. Bužek, N. Imoto, Phys. Rev. A **62**, 050302 (2000)
[16] W. Dür, Phys. Rev. A **63**, 020303(R) (2001)
[17] J. Mácha, arXiv quant-ph/0005115 (2000)
[18] R. Raussendorf, H. Briegel, Phys. Rev. Lett. **86**, 5188 (2001)