Additivity of Entanglement of Formation of Two Three-level-antisymmetric States

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Abstract

Quantum entanglement is the quantum information processing resource. Thus it is of importance to understand how much of entanglement particular quantum states have, and what kinds of laws entanglement and also transformation between entanglement states subject to. Therefore, it is essentialy important to use proper measures of entanglement which have nice properties. One of the major candidates of such measures is "entanglement of formation", and whether this measurement is additive or not is an important open problem. We aim at certain states so-called "antisymmetric states" for which the additivity are not solved as far as we know, and show the additivity for two of them.

Keywords: quantum entanglement, entanglement of formation, additivity of entanglement measures, antisymmetric states.

1 Introduction

Concerning the additivity of entanglement of formation, only a few results have been known. Vidal et al. [1] showed that additivity holds for some mixture of Bell states and other examples by reducing the argument of additivity of the Holevo capacity of so-called "entanglement breaking quantum channels" [2] and they are the non-trivial first examples. Matsumoto et al. [3] showed that additivity of entanglement of formation holds for a family of mixed states by utilizing the additivity of Holevo capacity for unital qubit channels [4], depolarizing channels and entanglement-breaking channels via Stinespring dilation [5].

In this poster it is shown that entanglement of formation is additive for tensor product of two three-dimensional bipartite antisymmetric states with a sketch of the proof. We proved by combination of elaborate calculations.

2 New additivity result

2.1 Antisymmetric states

Let us start with an introduction of our notations and concepts. $\mathcal{H}_-$ will stand for an antisymmetric Hilbert space, which is a subspace of a bipartite Hilbert space $\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$, where both $\mathcal{H}_A$ and $\mathcal{H}_B$ are $3-$dimensional Hilbert spaces, spanned by basic vectors $\{|i\rangle\}_{i=1}^3$. $\mathcal{H}_-$ is three-dimensional Hilbert space, spanned by states $\{|i,j\rangle\}_{ij=23,31,12}$, where the state $|i,j\rangle$ is defined as $\frac{|i\rangle|j\rangle-|j\rangle|i\rangle}{\sqrt{2}}$. The space $\mathcal{H}_-$ is called antisymmetric because by swapping the position of two qubits in any of its states $|\psi\rangle$ we get the state $-|\psi\rangle$. Let $\mathcal{H}_-^{\otimes n}$ be the tensor product of n copies of $\mathcal{H}_-$. These copies will be discriminated by the upper index as $\mathcal{H}_-^{(j)}$, for $j = 1 \dots n$. $\mathcal{H}_-^{(j)}$ will then be an antisymmetric subspace of $\mathcal{H}_A^{(j)} \otimes \mathcal{H}_B^{(j)}$.

2.2 The result and proof sketch

It has been shown in [1] that $E_f(\rho) = 1$ for any mixed state $\rho \in \mathcal{S}(\mathcal{H}_-)$. This result will play the key role in our proof. We prove now that :

Theorem.

$$E_f(\rho_1 \otimes \rho_2) = E_f(\rho_1) + E_f(\rho_2) \ (= 2) \tag{1}$$

for any $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{H}_-)$.

Proof. To prove this theorem, it is sufficient to show that

$$E_f(\rho_1 \otimes \rho_2) \geq 2 \tag{2}$$

since the subadditivity $E_f(\rho_1 \otimes \rho_2) \leq E_f(\rho_1) +$

$E_f(\rho_2) = 2$ is trivial. Indeed, it holds

$$\begin{aligned}
E_f\ (\rho_1 \otimes \rho_2) &= \inf \sum p_i E(|\psi_i\rangle\langle\psi_i|) \\
&\leq \inf \sum p_i^{(1)} p_i^{(2)} E(|\psi_i^{(1)}\rangle\langle\psi_i^{(1)}| \otimes |\psi_i^{(2)}\rangle\langle\psi_i^{(2)}|) \\
&= \inf \sum p_i^{(1)} E(|\psi_i^{(1)}\rangle\langle\psi_i^{(1)}|) \\
&\quad + \inf \sum p_i^{(2)} E(|\psi_i^{(2)}\rangle\langle\psi_i^{(2)}|) \\
&= E_f(\rho_1) + E_f(\rho_2)
\end{aligned} \tag{3}$$

where $(p_i^{(j)}, |\psi_i^{(j)}\rangle)$ are subject to the condition of $\rho_j = \sum_i p_i^{(j)} |\psi_i^{(j)}\rangle\langle\psi_i^{(j)}|$. To prove (2), we first show that

$$E(|\psi\rangle\langle\psi|) \geq 2, \text{ for any pure state } |\psi\rangle \in \mathcal{H}_-^{\otimes 2}. \tag{4}$$

Using the Schmidt decomposition, the state $|\psi\rangle$ can be decomposed as follows:

$$|\psi\rangle = \sum_{i=1}^{3} \sqrt{p_i}\, |\psi_i^{(1)}\rangle \otimes |\psi_i^{(2)}\rangle, \tag{5}$$

where $p_1, p_2, p_3 > 0, p_1 + p_2 + p_3 = 1$, and $\{|\psi_i^{(j)}\rangle\}_{i=1}^3$ is an orthonormal basis of the Hilbert space $\mathcal{H}_-^{(j)}$, for $j = 1, 2$. Note that this Schmidt decomposition is with respect to $\mathcal{H}_-^{(1)} : \mathcal{H}_-^{(2)}$, or, it could be said that with respect to $\left(\mathcal{H}_A^{(1)} \otimes \mathcal{H}_B^{(1)}\right) : \left(\mathcal{H}_A^{(2)} \otimes \mathcal{H}_B^{(2)}\right)$, not with respect to $\left(\mathcal{H}_A^{(1)} \otimes \mathcal{H}_A^{(2)}\right) : \left(\mathcal{H}_B^{(1)} \otimes \mathcal{H}_B^{(2)}\right)$, where ":" indicates how to separate the system into two subsystems for the decomposition.

First, we will use the following fact.

Lemma. *If $\{|\psi_i\rangle\}_{i=1}^3$ is an orthonormal basis of $\mathcal{H}_-$, then there exists an unitary operator U, acting on both $\mathcal{H}_A$ and $\mathcal{H}_B$, such that $U \otimes U$ maps the states $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ into the states $|2, 3\rangle, |3, 1\rangle, |1, 2\rangle$, respectively.*

Therefore, by this Lemma, there exist unitary operators $U^{(1)}, U^{(2)}$ such that

$$\begin{aligned}
&\left(U^{(1)} \otimes U^{(1)} \otimes U^{(2)} \otimes U^{(2)}\right)|\psi\rangle \\
&\quad = \sum_{\substack{i,j \\ ij=23,31,12}} \sqrt{p_{ij}}\, |i, j\rangle \otimes |i, j\rangle \ =:\ |\psi'\rangle,
\end{aligned} \tag{6}$$

where $p_{23} := p_1,\ p_{31} := p_2,\ p_{12} := p_3$.

As is written in the following, we use the following fact.

Lemma.

$$E(|\psi'\rangle\langle\psi'|) \geq 2, \quad \textit{if} \quad \left\{ \begin{array}{l} p_{23}, p_{31}, p_{12} \geq 0 \\ p_{23} + p_{31} + p_{12} = 1 \end{array} \right. . \tag{7}$$

(We proved this lemma by solving a cubic equation and bounding the Shannon entropy function with polynomial functions.) Local unitary operators do not change von Neumann reduced entropy, and therefore $E(|\psi\rangle\langle\psi|) = E(|\psi'\rangle\langle\psi'|) \geq 2$. That is, the claim (4) is proven.

We are now almost done. Indeed, the entanglement of formation is defined as

$$E_f(\rho) = \inf_{[(p_i,\psi_i)]_i \in \Delta(\rho)} \sum_i p_i E(|\psi_i\rangle\langle\psi_i|) \tag{8}$$

where

$$\Delta(\rho) = \left\{ [(p_i, \psi_i)]_i \;\middle|\; \begin{array}{l} \sum_i p_i = 1, p_i > 0 \forall i \\ \sum_i p_i |\psi_i\rangle\langle\psi_i| = \rho, \langle\psi_i|\psi_i\rangle = 1 \forall i \end{array} \right\}$$

and it is known that all $|\psi_i\rangle$ induced from $\Delta(\rho)$ satisfy $|\psi_i\rangle \in \mathrm{Range}(\rho)$, where $\mathrm{Range}(\rho)$ is sometimes called the image space of the matrix ρ, which is the set of $\rho|\psi\rangle$ with $|\psi\rangle$ running over the domain of ρ. Hence

$$E_f(\rho) \geq \inf \left\{ E(|\psi\rangle\langle\psi|) \;\middle|\; |\psi\rangle \in \mathrm{Range}(\rho), \langle\psi|\psi\rangle = 1 \right\}. \tag{9}$$

Since $\rho_1 \otimes \rho_2 \in \mathcal{S}(\mathcal{H}_-^{\otimes 2})$, $\mathrm{Range}(\rho_1 \otimes \rho_2) \subseteq \mathcal{H}_-^{\otimes 2}$, henceforth (2) is proven. Therefore (1) have been shown. □

3 Conclusions and discussion

Additivity of the entanglement of formation for two three-dimensional bipartite antisymmetric states has been proven in this paper. The next goal could be to prove additivity for more than two antisymmetric states. Perhaps the proof can utilize the value of lower bound of the reduced von Neumann entropy. Of course, the main goal is to show that entanglement of formation is additive, in general. However, this seems to be a very hard task.

References

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