Concurrence hierarchy: A measurement of entanglement

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We define the concurrence hierarchy as $d-1$ independent invariants under local unitary transformations in d-level quantum system. The first one is the original concurrence defined by Wootters et al [1,2] in 2-level quantum system and generalized to d-level pure quantum states case. We propose to use this concurrence hierarchy as measurement of entanglement. This measurement does not increase under local quantum operations and classical communication (LOCC).

For 2-level bipartite quantum state, Wootters [1,2] proposed to use concurrence as the measure of entanglement which is monotonically increasing respect to entanglement of formation, a widely accepted meansure of entanglement [3,4]. Because concurrence provide a measure of entanglement in 2-level system, it is worth generalizing concurrence to higher dimension. There are several proposals for the case of pure states [6,7,5,8,10,11] by using different methods. Uhlmann generalized the concurrence by considering arbitrary conjugations acting on arbitrary Hilbert spaces [6]. Other groups results are almost the same, the concurrence is defined as the quantity $C(\Phi)=\sqrt{2[1-Tr(\rho_A^2)]}$, where $\rho_A=Tr_B(|\Phi\rangle\langle\Phi|)$ is the reduced density operator. These two generalizations have a close relation pointed out by Wootters [9] and are essentially lead to the same result.

Nielsen found a remarkable result of classifying the entanglement by the majorization scheme [14]. For convenience, we use the same notations as that of Ref. [15] and Nielsen. The elements of vectors $x=\{x_0^{\downarrow},\cdots,x_{d-1}^{\downarrow}\}$ and $y=\{y_0^{\downarrow},\cdots,y_{d-1}^{\downarrow}\}$ are ordered in decreasing order. We say that x is majorized by y, $x\prec y$, if $\sum_{j=0}^k x_j^{\downarrow}\le\sum_{j=0}^k y_j^{\downarrow}, k=0,\cdots,d-1$ and the equality holds when $k=d-1$. Suppose a bipartite pure state $|\Phi\rangle$ shared by A and B, $\lambda_\Phi=\{\lambda_0^{\downarrow},\cdots,\lambda_{d-1}^{\downarrow}\}$ denotes the vector of eigenvalues of the reduced density operator $\rho_A=Tr_B(|\Phi\rangle\langle\Phi|)$ in decreasing order. In other words $\lambda_j^{\downarrow}, j=0,\cdots,d-1$ are square of singular values of matrix Λ.

Theorem by Nielsen [14]: $|\Psi\rangle$ *transforms to* $|\Phi\rangle$ *using LOCC if and only if* λ_Ψ *is majorized by* λ_Φ,

$$|\Psi\rangle\to|\Phi\rangle\quad \text{iff}\quad \lambda_\Psi\prec\lambda_\Phi. \tag{1}$$

Nielsen theorem provides a necessary and sufficient condition in transforming entangled bipartite pure states by LOCC. And Nielsen theorem can be applied to study entanglement quantification and entanglement manipulation.

We know the following statesments are equivalent:

(i) $x\prec y$.
(ii) x is obtained from y by a finite number of T-transforms.
(iii) $x=Ay$ *for some doubly stochastic matrix A.*
(iv) $x=\sum_j p_jP_jy$ *for some probability distribution* p_j *and permutation matrices* P_j.
where $Ty=(y_1,\cdots,y_{j-1},ty_j+(1-t)y_k,\cdots,(1-t)y_j+ty_k,\cdots,y_d)$.

By using (ii), where the number of T-transforms is bounded by $d-1$, Nielsen proposed an algorithm to transfer entangled bipartite pure states by LOCC [14], the entanglement transformation requires $d-1$ bits of classical communication. Later on, by using (iv), some other algorithms were proposed, and the protocol requires 2log$d$ bits of commmunication [16,17]. The tight lower bound on classical communication cost of entanglement dilution was proposed and proved in [18,19].

Nielsen theorem also let us find some Schur-concave functions which can be used as entanglement measures. In this paper, we propose to use the concurrence hierarchy to quantify the entanglement for d-dimension which can be proved to be Schur-concave functions according to majorization scheme. We restrict ourself to $C^d\otimes C^d$ bipartite pure state. A general bipartite pure state in $C^d\otimes C^d$ can be written as $|\Phi\rangle=\sum_{i,j=0}^{d-1}\alpha_{ij}|ij\rangle$, with normalization $\sum_{ij}\alpha_{ij}\alpha_{ij}^*=1$. We define a matrix Λ with entries $\Lambda_{ij}=\alpha_{ij}$. The reduced density matrix can be denoted as $\rho_A=Tr_B\rho=\Lambda\Lambda^\dagger$. Under a local unitary transformation $U\otimes V$, the matrix Λ is changed to $\Lambda\to U^t\Lambda V$, where the superindex t represents transposition. And the redeced density operator thus is transformed to $\rho_A\to(U^t\Lambda V)(V^\dagger\Lambda^\dagger U^{t\dagger})=U^t\Lambda\Lambda^\dagger U^{t\dagger}$. In 2-dimension, it was point out by Linden and Popescu [13], there is one no-trivial invariant under local unitary transformations $I=Tr(\Lambda\Lambda^\dagger)^2$. In general d-dimension, it was pointed out by Albeverio and Fei that there are $d-1$ independent invariants under local unitary transformations $I_k=Tr(\Lambda\Lambda^\dagger)^{k+1}$. When $k=0$, it is just the normalization equation $I_0 = \sum_{ij} \alpha_{ij}\alpha_{ij}^* = 1$. For $k = 1, \cdots, d-1$, I_k are $d-1$ independent invariants under local unitary transformations.

Next, we give our precise definition of concurrence hierarchy. Suppose a bipartite pure state shared by A and B, $\lambda_\Phi = \{\lambda_0^\downarrow, \cdots, \lambda_{d-1}^\downarrow\}$ denotes the vector of eigenvalues of the reduced density operator $\rho_A = Tr_B(|\Phi\rangle\langle\Phi|)$ in decreasing order.

Definition: *The concurrence hierarchy of the state $|\Phi\rangle$ is defined as*

$$C_k(\Phi) = \sum_{0 \le i_0 < i_1 < \cdots < i_k \le (d-1)} \lambda_{i_0}^\downarrow \lambda_{i_1}^\downarrow \cdots \lambda_{i_k}^\downarrow, \qquad k = 1, 2, \cdots, d-1. \tag{2}$$

We propose to use this concurrence hierarchy to quantify the entanglement of the state $|\Phi\rangle$.

The first level concurrence is trivial since it is just the normalization condition $C_1(\Phi) = \sum_{i=0}^{d-1} \lambda_i^\downarrow = 1$. The two level concurrence is the d-dimension generalization of concurrence proposed by Rungta et al [7] and Albeverio et al [8] and others [5,10] up to a whole factor. In 2-dimension, there are just one non-trivial concurrence which is the original concurrence proposed by Wootters et al [1,2]. In d-dimension, the concurrence hierarchy consists of $d-1$ independent non-trivial concurrences. This concurrence hierarchy is invariant under local unitary transformations and can be represented in terms of invariants $I_k = Tr(AA^\dagger)^{k+1}$ [8]. It should be noted that a similar idea as this paper was also proposed by Sinolecka et al [12].

According to some results in linear algebra, see for example Ref. [15], the concurrence hierarchy $C_k(\Phi)$ equal to the sums of the k-by-k principal minors of reduced density operator $\Lambda\Lambda^\dagger$. And it is known that these quantities are invariant under unitary transformations $U\Lambda\Lambda^\dagger U^\dagger$. This leads straightforward to the result that for a bipartite pure state, the concurrence hierarchy $C_k(\Phi)$ are invariant under local unitary transformations. For convenience, we adopt the same notations as that of Ref. [15]. Let $\beta, \gamma \subseteq \{0, ..., d-1\}$ be index sets, each of cardinality k, $k = 1, \cdots, d$. According to Cauchy-Binet formula, we have the following relations:

$$C_k(\Phi) = \sum_\beta \det \rho_A(\beta, \beta) = \sum_\beta \sum_\gamma |\det \Lambda(\beta, \gamma)|^2, \tag{3}$$

where we use the relation $\rho_A = \Lambda\Lambda^\dagger$, and the notation $\det \Lambda(\beta, \gamma)$ means the determinant of submatrix Λ with row and column index sets β and γ. When the cardinality $k = 2$, we recover some known results. So, we do not need to calculate the eigenvalues of the reduced density operator to find the concurrence hierarchy, we can calculate the concurrence hierarchy directly by summing the determinants of all k-by-k submatrices of Λ.

Using the concurrences in the hierarchy is more powerful than the case that only one concurrence is used. However, it is not complete though the hierarchy consists of $d-1$ independent invariants.

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