A Theory of Matrices Formally Identical to Algorithmic Information Theory

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Abstract

There are two equivalent ways to define the Kolmogorov complexity $H(s)$ of a given classical finite binary string s. In the standard way, $H(s)$ is defined as the length of the shortest input for a universal Turing machine to output s. In another way, we first construct what is called the "universal probability" μ, and define $H(s)$ as $-\log_2 \mu(s)$ without using the concept of program-size. We generalize the universal probability to a matrix-valued function, and define a matrix-valued Kolmogorov complexity. In the same context as algorithmic information theory, i.e., a theory of program-size complexity which has precisely the formal properties of classical information theory, we define matrix-valued joint entropy, conditional entropy, and mutual information. Then we prove several relations which appear in classical information theory, except for subadditivity. We discuss prospects in thinking of the matrix-valued universal probability as a POVM.

Key words: algorithmic information theory, Kolmogorov complexity, universal probability, POVM

Algorithmic information theory is a theory of program-size complexity which has precisely the formal properties of classical information theory. In algorithmic information theory, the program-size complexity (or Kolmogorov complexity) $H(s)$ of a finite binary sequence s is defined as the length of the shortest binary input for a universal algorithm U to output s. The concept of program-size complexity plays an important role in characterizing the randomness of an infinite binary sequence.

Our aim is to generalize algorithmic information theory to quantum region. There were already several proposals for such a generalization.[5, 1, 4] In each of [5] and [1], the quantum Kolmogorov complexity of qubits was defined by using the universal quantum Turing machine as a decoding universal algorithm U, while there is a difference between [5] and [1] with respect to program which is allowed as an input to U. That is, [5] can only allow classical bits as an input, whereas [1] can allow any qubits. Actually, there are two equivalent ways to define the classical Kolmogorov complexity $H(s)$. In the standard way, $H(s)$ is defined as the length of the shortest input for a universal algorithm to output a classical finite binary sequence s. [5]

and [1] followed this standard way. In another way, we first construct the so-called universal probability μ, and define $H(s)$ as $-\log_2 \mu(s)$ without using the concept of program-size. [4] followed this another way, and generalized the universal probability to a matrix with the trace less than or equal to one. Then [4] proposed to regard this generalized universal probability as the density matrix of a quantum system. The trace of a density matrix has to be equal to one in general. However, the trace of this generalized universal probability cannot be equal to one because of its universality. So it would seem that there is a conceptual contradiction in the interpretation of this generalized universal probability as a density matrix. In quantum mechanics, what is represented by a matrix is either a quantum state or a measurement operator. In this presentation, we generalize the universal probability to a matrix-valued function in different way from [4], and propose to regards it as a POVM. We also define a matrix-valued Kolmogorov complexity by using this matrix-valued universal probability. In the same context as algorithmic information theory, we define matrix-valued joint entropy, conditional entropy, and mutual information. Then we can prove the relations which appear in classical information theory, except for subadditivity. This exception seems to justify our interpretation of the matrix-valued universal probability as a POVM. We also discuss other prospects of this interpretation in the presentation.

References

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