A remark on concavity of the function appearing in quantum reliability function

S. Furuichi[1], K.Yanagi[2] and K.Kuriyama[2]
[1]Tokyo University of Science in Yamaguchi, Onoda city, Yamaguchi, 756-0884, Japan
[2]Yamaguchi University, Ube city, Yamaguchi, 755-8611, Japan

In quantum information theory, it is important to study the properties of the function $E_q(\pi, s)$, which will be defined in the below, appearing in the lower bound with respect to the random coding in the reliability function for general quantum states. In classical information theory [1], the random coding exponent $E_r^c(R)$, the lower bound of the reliability function, is defined by

$$E_r^c(R) = \max_{p,s}\left[-sR + E_c(p,s)\right].$$

As for the function $E_c(p, s)$, it is well-known the following properties [1].

(a) $E_c(p, 0) = 0$.

(b) $\frac{\partial E_c(p,s)}{\partial s}|_{s=0} = I(X;Y)$, where $I(X;Y)$ presents the classical mutual information.

(c) $E_c(p, s) > 0$ $(0 < s \leq 1)$. $E_c(p, s) < 0$ $(-1 < s \leq 0)$.

(d) $\frac{\partial E_c(p,s)}{\partial s} > 0$, $(-1 < s \leq 1)$.

(e) $\frac{\partial^2 E_c(p,s)}{\partial s^2} \leq 0$, $(-1 < s \leq 1)$.

In quantum case, the corresponding properties to (a),(b),(c) and (d) have been shown in [3, 5]. Also the concavity of the function $E_q(\pi, s)$ is shown in the case when the signal states are pure [4], and when the expurgation method is adopted [5]. However, for general signal states, the concavity of the function $E_q(\pi, s)$ which corresponds to (e) in the above has remained as an open question [3, 5].

The reliability function of classical-quantum channel is defined by

$$E(R) \equiv -\liminf_{n\to\infty} \frac{1}{n}\log P_e(2^{nR}, n), \quad 0 < R < C, \tag{1}$$

where C is a classical-quantum capacity, R is a transmission rate $R = \frac{\log_2 M}{n}$ (n and M represent the number of the code words (or dimension of the Hilbert spaces) in input system and output system, respectively), $P_e(M, n)$ can be taken any minimal error probabilities of $\min_{\mathcal{W},\mathsf{X}} \bar{P}(\mathcal{W}, \mathsf{X})$ or $\min_{\mathcal{W},\mathsf{X}} P_{\max}(\mathcal{W}, \mathsf{X})$. These error probabilities are defined by

$$\begin{aligned}\bar{P}(\mathcal{W}, \mathsf{X}) &= \frac{1}{M}\sum_{j=1}^{M} P_j(\mathcal{W}, \mathsf{X}),\\ P_{\max}(\mathcal{W}, \mathsf{X}) &= \max_{1\leq j\leq M} P_j(\mathcal{W}, \mathsf{X}),\end{aligned}$$

where

$$P_j(\mathcal{W}, \mathrm{X}) = 1 - \mathrm{Tr} S_{w^j} X_j$$

is the usual error probability associated with the positive operator valued measurement X = $\{X_j\}$ satisfying $\sum_{j=1}^{M} X_j \leq I$. Here we note S_{w^j} represents the density operator corresponding to the code word w^j choosen from the code(blook) $\mathcal{W} = \{w^1, w^2, \cdots, w^M\}$. For details, see [2, 3, 5].

The lower bound for the quantum reliability function defined in Eq.(1), when we use random coding, is given by

$$E(R) \geq E_r^q(R) \equiv \max_{\pi} \sup_{0<s\leq 1} \left[E_q(\pi, s) - sR\right],$$

where $\pi = \{\pi_1, \pi_2, \cdots, \pi_a\}$ is a priori probability distribution satisfying $\sum_{i=1}^{a} \pi_i = 1$ and

$$\begin{aligned} E_q(\pi, s) &= -\log G(s), \\ G(s) &= \mathrm{Tr}\left[A(s)^{1+s}\right], \\ A(s) &= \sum_{i=1}^{a} \pi_i S_i^{\frac{1}{1+s}}, \end{aligned}$$

where each S_i is density operator which corresponds to the output state of the classical-quantum channel $i \to S_i$ from the set of the input alphabet $A = \{1, 2, \cdots, a\}$ to the set of the output quantum states in the Hilbert space $\mathcal{H}$. Then we have the following.

Proposition 0.1 For any real number s $(-1 < s \leq 1)$, density operators $S_i (i = 1, \cdots, a)$ and a priori probability $\pi = \{\pi_i\}_{i=1}^{a}$, if the inequality

$$\left(\sum_{i=1}^{a} \pi_i S_i^{\frac{1}{1+s}}\right)^{\frac{1}{2}} \left\{\sum_{j=1}^{a} \pi_j S_j^{\frac{1}{1+s}} \left(\log S_j^{\frac{1}{1+s}}\right)^2\right\} \left(\sum_{i=1}^{a} \pi_i S_i^{\frac{1}{1+s}}\right)^{\frac{1}{2}} \geq \left\{\sum_{i=1}^{a} \pi_i H\left(S_i^{\frac{1}{1+s}}\right)\right\}^2$$

holds, then the function

$$E_q(\pi, s) = -\log\left[\mathrm{Tr}\left\{\left(\sum_{i=1}^{a} \pi_i S_i^{\frac{1}{1+s}}\right)^{1+s}\right\}\right]$$

is concave in s.

References

[1] R.G.Gallager, Information theory and reliable communication, John Wiley and Sons,1968.

[2] A.S.Holevo, The capacity of quantum channel with general signal states, IEEE.Trans.IT, Vol.44, 269(1998).

[3] T.Ogawa and H.Nagaoka, Strong converse to the quantum channel coding theorem, IEEE.Trans.IT, Vol.45, 2486(1999).

[4] M.V.Burnashev and A.S.Holevo, On reliability function of quantum communication channel, quant-ph/9703013.

[5] A.S.Holevo, Reliability function of general classical-quantum channel, IEEE.Trans.IT, Vol.46, 2256(2000).