# Locally-induced global disorder in quantum composite systems

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It is investigated how local quantum operations in bipartite quantum systems induce a global order or disorder. It is proven that if the global order is never recovered by any local operations of the form $\Lambda_A \otimes \mathbf{I}_B$ then the initial state $\rho$ is necessarily a distillable entangled state satisfying $S(\rho_B) \geq S(\rho)$, where $S(\rho)$ denotes the von Neumann entropy and $\rho_B = \mathrm{Tr}_A \rho$. Furthermore, it is also proven that if a state $\rho$ is undistillable (separable or bound entangled) then the minimum von Neumann entropy of the locally disturbed state $\min_{\Lambda_A} S((\Lambda_A \otimes \mathbf{I}_B)\rho)$ is bounded from below by $S(\rho_B)$.

**Key words:** Quantum entanglement, Local quantum operation, von Neumann entropy

Most of the basic tasks in quantum information processing are boiled down to manipulations of quantum entanglement. It is, therefore, of crucial importance to understand the nature of quantum entanglement. The quantitative properties of entanglement should be in principle characterized by several entanglement measures which are still intensively investigated [1], yet there are still other viewpoints to see many aspects of quantum entanglement such as the fidelity of quantum teleportation [2], the capacity of dense coding [3], etc. In this note, I consider density matrices or quantum states on Hilbert space $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ and investigate how local quantum operations (on the local system $A$ or $B$) induce an order or a disorder to the global $(A + B)$ system. A quantum operation is represented by a completely positive and trace preserving (CPTP) map $\Lambda$, which takes the form [4]

$$\Lambda(\rho) = \sum_{i=1}^{d^2} V_i \rho V_i^\dagger, \tag{1}$$

where $d$ is the dimension of the Hilbert space on which $\rho$ acts. The trace preserving condition $\mathrm{Tr}\Lambda(\rho) = \mathrm{Tr}\rho$ is equivalent to the following equality.

$$\sum_{i=1}^{d^2} V_i^\dagger V_i = \mathbf{I}. \tag{2}$$

Eq. (1) is known as the operator-sum representation of the quantum operation $\Lambda$. Hereafter, it is said that a density matrix $\rho_1$ is more disordered than a density matrix $\rho_2$ when $S(\rho_1) \geq S(\rho_2)$, where $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$ denotes the von Neumann entropy. For example, if a $d$-dimensional quantum state $\rho$ is a maximally disordered state, i.e., $\rho = \mathbf{I}_d/d$, then $S(\rho) = \log_2 d$, which is the maximal value of the von Neumann entropy. Conversely, if $\rho$ is a pure state, then $S(\rho) = 0$. In this sense, a pure state is a completely ordered state. Separable states satisfy $S(\rho_B) \leq S(\rho)$, where $\rho_B = \mathrm{Tr}_A \rho$ is the reduced density matrix on the system $B$. That is, "Separable states are more disordered globally than locally" as summarized in [5]. This is the direct consequence of Theorem 1 of [5]. It is also derived from the following reduction criterion [6–8].

**Lemma 1** *If $\rho$ is not distillable, then $\mathbf{I}_A \otimes \rho_B - \rho$ is a non-negative operator;*

$$\mathbf{I}_A \otimes \rho_B \geq \rho. \tag{3}$$

Noting the operator monotonicity of the logarithm [9], i.e., if $A \geq B$, then $\ln A \geq \ln B$, it is easy to show that Eq. (3) implies the entropic inequality $S(\rho_B) \leq S(\rho)$. Conversely, if a state $\rho$ is more disordered locally than globally, i.e., $S(\rho_B) > S(\rho)$, then $\rho$ is distillable; it is neither separable nor bound entangled [10].

In the following, I show that a state in which the global order is never recovered by any local quantum operations is necessarily entangled. More precisely, I prove the following theorem.

**Theorem 1** *If $S(\rho) \leq S((\Lambda_A \otimes \mathbf{I}_B)\rho)$ holds for every CPTP map $\Lambda_A$, then $\rho$ is a distillable entangled state satisfying $S(\rho_B) \geq S(\rho)$.*

**Proof.** Suppose that $S(\rho_B) < S(\rho)$ holds. Let us consider the following CPTP map in the operator-sum representation with $d_A$ operation elements $V_i$:

$$\Lambda_A(\sigma_A) = \sum_{i=1}^{d_A} V_i \sigma_A V_i^\dagger, \tag{4}$$

where $\sigma_A$ is any density matrix on the system $A$. Here

$$V_i = U^{(i)} \left|i\right\rangle_A \left\langle i\right|, \quad (i = 1, 2, \cdots, d_A), \tag{5}$$

with $\left|i\right\rangle_A$ are orthogonal basis for the system $A$ and $U^{(i)}$ are $d_A \times d_A$ unitary matrices of the form:

$$U^{(i)} = \begin{array}{c} \begin{array}{ccccccc} 1 & \cdots & i-1 & i & i+1 & \cdots & d_A \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & 0 & & & \\ & * & & \vdots & & * & \\ & & & 0 & & & \end{pmatrix} \end{array}. \tag{6}$$

It is easy to check that the trace preserving condition $\sum_{i=1}^{d} V_i^\dagger V_i = \mathbf{I}_A$ holds. The explicit calculation yields

$$\sum_{i=1}^{d_A} (V_i \otimes \mathbf{I}_B)\rho(V_i^\dagger \otimes \mathbf{I}_B) = |1\rangle_A \langle 1| \otimes \rho_B. \quad (7)$$

This state is obviously separable. The von Neumann entropy for this state is calculated as

$$S((\Lambda_A \otimes \mathbf{I}_B)\rho) = S\left(|1\rangle_A \langle 1| \otimes \rho_B\right) = S(\rho_B) < S(\rho). \quad (8)$$

This completes the proof.

Theorem 1 is the main result of this note. Note that the condition of Theorem 1 is not a necessary condition for entanglement; even if a state $\rho$ is an entangled state, it does not necessarily satisfy the condition of Theorem 1. Unfortunately, the converse of Theorem 1 does not hold in general; many counter-examples on $\mathbb{C}^3 \otimes \mathbb{C}^3$ systems have been found numerically. On the other hand, numerical searches on $\mathbb{C}^2 \otimes \mathbb{C}^2$ systems suggest that the converse holds for two qubit systems; $\max\{0, S(\rho_B) - S(\rho)\}$ is weakly monotonic under quantum operations of the form $\Lambda_A \otimes \mathbf{I}_B$. This is also pointed in [11], although it is still unproven.

In the proof of Theorem 1, I have constructed a CPTP map $\Lambda_A$ reducing the disorder of the global system $\rho$ to $S(\rho_B)$. Is it possible to reduce the disorder further? The answer is negative for undistillable states.

**Theorem 2** *If $\rho$ is an undistillable (separable or bound entangled) state, then*

$$\min_{\Lambda_A} S((\Lambda_A \otimes \mathbf{I}_B)\rho) = S(\rho_B), \quad (9)$$

*where the minimum is taken over all possible quantum operations.*

**Proof.** Because local operations cannot change an undistillable state to a distillable one, the state $(\Lambda_A \otimes \mathbf{I}_B)\rho$ is also an undistillable state. Noting $\mathrm{Tr}_A((\Lambda_A \otimes \mathbf{I}_B)\rho) = \rho_B$ and the fact that any undistillable state $\sigma$ satisfies $S(\sigma) \geq S(\sigma_B)$, we obtain $S((\Lambda_A \otimes \mathbf{I}_B)\rho) \geq S(\rho_B)$. The equality holds for the quantum operation described in the proof of Theorem 1. This completes the proof.

The numerical work on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^3 \otimes \mathbb{C}^3$ systems suggests that Theorem 2 holds for any state (undistillable or distillable) satisfying $S(\rho_B) < S(\rho)$. Therefore, I conjecture the following.

**Conjecture 1** *If $\rho$ is a state such that $S(\rho_B) < S(\rho)$, then*

$$\min_{\Lambda_A} S((\Lambda_A \otimes \mathbf{I}_B)\rho) = S(\rho_B), \quad (10)$$

*where the minimum is taken over all possible quantum operations.*

In view of the optimal capacity of dense coding [12,13],

$$\chi^*(\rho) = \log_2 d_A + S(\rho_B) - S(\rho), \quad (11)$$

Conjecture 1 leads a reasonable consequence; If $\chi^*(\rho)$ is already below $\log_2 d_A$ - the classically achievable capacity, the sender cannot boost up the capacity above the classical limit by any local quantum operations.

In this note, the von Neumann entropy is adopted as a measure of the disorder of states. Yet another way to characterize the disorder of states exists. A notable device is the theory of majorization [14], which is more suitable to capture the notion of disorder as argued in [5]. Therefore, it is interesting to ask whether Theorem 1 or Theorem 2 can be rephrased in terms of majorization. This remains an open question for further research.

This work was supported by CREST of Japan Science and Technology Corporation (JST).

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