Convex-roof extended negativity as an entanglement measure for bipartite quantum systems

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Abstract. We extend the concept of the negativity, a good measure of entanglement for bipartite pure states, to mixed states by means of the convex-roof extension. We show that the measure does not increase under local quantum operations and classical communication, and derive explicit formulae for the entanglement measure of isotropic states and Werner states, applying the formalism presented by Vollbrecht and Werner [Phys. Rev. A **64**, 062307 (2001)].

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Quantum information processing essentially depends on several quantum mechanical phenomena, among which entanglement has been considered as one of the most crucial features. There are two important problems for entanglement. One is to find a method to determine whether a given state in an arbitrary dimensional quantum system is separable or not, and the other is to define the best measure quantifying an amount of entanglement of a given state. In order to solve these problems, various criteria for separability and not a few measures of entanglement have been proposed in recent years [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Although the perfect solutions for the problems have not yet been obtained, quite a good criterion for separability, called the *positive partial transposition* (PPT) criterion, was suggested by Peres [1] and Horodecki *et al.* [3], and an entanglement measure was naturally derived from the PPT criterion [12, 15, 16]. The measure is called the *negativity* [17, 12], and is defined by

$$\mathcal{N}(\rho) = \frac{\|\rho^{T_B}\|_1 - 1}{d-1}, \tag{1}$$

where ρ^{T_B} is the partial transpose of a state ρ in $d \otimes d'$ ($d \le d'$) quantum system and $\|\cdot\|_1$ is the trace norm.

However, although the positivity of the partial transpose is a necessary and sufficient condition for nondistillability in $2 \otimes n$ quantum system [4, 18], there exist entangled states with PPT in any bipartite system except in $2 \otimes 2$ and $2 \otimes 3$ quantum systems [4, 8], that is, there exist entangled states whose negativity are not positive. Hence, it is not sufficient for the negativity to be a good measure of entanglement even in $2 \otimes n$ quantum system.

We now consider the negativity of pure states in $d \otimes d'$ ($d \le d'$) quantum system, $\mathcal{H}_A \otimes \mathcal{H}_B$. By the Schmidt decomposition theorem, a given pure state $|\Psi\rangle$ can be written as

$$|\Psi\rangle = \sum_{j=0}^{d-1} \sqrt{\mu_j} |a_j b_j\rangle = U_A \otimes U_B |\Phi\rangle, \tag{2}$$

where $\sqrt{\mu_j}$ are the Schmidt coefficients, U_A and U_B are unitary operators defined by $U_A|j\rangle = |a_j\rangle$ and $U_B|j\rangle = |b_j\rangle$ respectively, and

$$|\Phi\rangle = \sum_{j=0}^{d-1} \sqrt{\mu_j} |jj\rangle. \tag{3}$$

Let $\left|\Psi_{ij}^{\pm}\right\rangle = \left(|ij\rangle \pm |ji\rangle\right)/\sqrt{2}$. Then since the partial transpose of $|\Phi\rangle\langle\Phi|$ is

$$\begin{aligned} |\Phi\rangle\langle\Phi|^{T_B} &= \sum_{k=0}^{d-1} \mu_k |kk\rangle\langle kk| + \sum_{i<j} \sqrt{\mu_i\mu_j} \left|\Psi_{ij}^{+}\right\rangle\left\langle\Psi_{ij}^{+}\right| \\ &\quad + \sum_{i<j} \left(-\sqrt{\mu_i\mu_j}\right) \left|\Psi_{ij}^{-}\right\rangle\left\langle\Psi_{ij}^{-}\right|, \end{aligned} \tag{4}$$

we have

$$\begin{aligned} \mathcal{N}(|\Psi\rangle\langle\Psi|) &= \mathcal{N}(|\Phi\rangle\langle\Phi|) \\ &= \frac{2}{d-1} \sum_{i<j} \sqrt{\mu_i\mu_j} \\ &\equiv \mathcal{N}_{\mathrm{p}}(\vec{\mu}), \end{aligned} \tag{5}$$

where $\vec{\mu} = (\sqrt{\mu_0}, \sqrt{\mu_1}, \ldots, \sqrt{\mu_{d-1}})$ is the Schmidt vector. We note that $\mathcal{N}_{\mathrm{p}}(\vec{\mu}) = 0$ if and only if $|\Psi\rangle$ is separable, and that $\mathcal{N}_{\mathrm{p}}((1,1,\ldots,1)/\sqrt{d}) = 1$. Thus $\mathcal{N}_{\mathrm{p}}$ can be a measure of entanglement for bipartite pure states in any dimensional quantum system, and can be extended to mixed states ρ by means of the convex roof,

$$\mathcal{N}_{\mathrm{m}}(\rho) \equiv \min_{\sum_k p_k |\Psi_k\rangle\langle\Psi_k| = \rho} \sum_k p_k \mathcal{N}_{\mathrm{p}}(\vec{\mu}_k), \tag{6}$$

where $\vec{\mu}_k$ is the Schmidt vector of $|\Psi_k\rangle$. The extended measure $\mathcal{N}_{\mathrm{m}}$ is called the *convex-roof extended negativity* (CREN). Then we can readily show that $\mathcal{N}_{\mathrm{m}}(\rho) = 0$ if and only if ρ is separable. This implies that the CREN can recognize the difference between separability and

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bound entanglement, which may not be done by the original negativity. We can also show that $\mathcal{N}_{\rm m}(\rho) \geq \mathcal{N}(\rho)$, by the convexity of the original negativity $\mathcal{N}$ [12]. In $2 \otimes 2$ quantum system, it follows from a straightforward calculation that the CREN $\mathcal{N}_{\rm m}$ is equivalent to the Wootters's concurrence [7, 11] since $\mathcal{N}_{\rm p}(\vec{\mu}) = 2\sqrt{\mu_0\mu_1} = |\langle\Psi|\tilde{\Psi}\rangle| = C(|\Psi\rangle)$, where $\vec{\mu}$ is the Schmidt vector of $|\Psi\rangle$, $|\tilde{\Psi}\rangle = \sigma_y \otimes \sigma_y |\Psi^*\rangle$, and C is the Wootters's concurrence.

Monotonicity of entanglement under local quantum operations and classical communication (LOCC) is considered as one of natural requirements which good measures of entanglement must hold. Vidal [19] gave a nice recipe for building entanglement monotones in bipartite quantum system by showing that the convex-roof extension of a pure-state measure E satisfying the following two conditions is an entanglement monotone: (i) For a reduced density matrix $\rho_A = {\rm tr}_B|\Psi\rangle\langle\Psi|$ of a pure state $|\Psi\rangle$, the function f on the space of density matrices defined by $f(\rho_A) = E(|\Psi\rangle)$ is invariant under unitary operations, that is, for any unitary operator U

$$f(U\rho_A U^\dagger) = f(\rho_A). \tag{7}$$

(ii) The function f is concave, that is, for any density matrices ρ_1, ρ_2, and any $\lambda \in [0,1]$,

$$f(\lambda\rho_1 + (1-\lambda)\rho_2) \geq \lambda f(\rho_1) + (1-\lambda) f(\rho_2). \tag{8}$$

In this paper, we show that the CREN is an entanglement monotone, by verifying that $\mathcal{N}_{\rm p}$ satisfies the above conditions.

Even though it is generally not so easy to calculate the value of the convex-roof extension of a pure-state measure, we can simplify the computation of entanglement measures for states that are invariant under a group of local symmetries [12, 14, 20, 21], such as isotropic states [9] and Werner states [22].

In this paper, we derive explicit formulae for the CREN of isotropic states and Werner states, exploiting the formalism presented by Vollbrecht and Werner [21]. This formalism originated from the method of Terhal and Vollbrecht [20], who gave an exact formula for the entanglement of formation for isotropic states, and a subsequent work by Rungta and Caves [14] recently provided explicit expressions for the concurrence-based entanglement measures of isotropic states. These computational results imply that the newly defined measure, CREN, is an entanglement measure not only to show the difference between separability and bound entanglement, but also to be computed as well as other convex-roof extended measures of entanglement.

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