Multipartite entanglement in $2 \times 2 \times$ n quantum systems

Akimasa Miyake[1,*] and Frank Verstraete[2,†]

[1] *Department of Physics, Graduate School of Science, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan*
[2] *Max-Planck Institut für Quantenoptik, Hans-Kopfermann Str. 1, Garching, D-85748, Germany*

We classify multipartite entangled states in the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$ $(n \geq 4)$, including the 4-qubit system distributed over 3 parties, under local filtering operations. We show that there exist nine essentially different classes of states, giving rise to a five-graded partially ordered structure. In particular, all $2 \times 2 \times n$-states can be deterministically prepared from one maximally entangled state, and some applications like the entanglement swapping are discussed.

Keywords: multipartite entanglement, stochastic LOCC

Entanglement is the key ingredient of all applications in the field of quantum information. Due to its nonlocal character, entanglement is expected to be especially valuable in the context of many parties. Despite many efforts however, it has been proven exceedingly hard to get insight into the structure of multipartite entanglement. Still, the motivation of our work is as follows. In the bipartite setting, the Einstein-Podolsky-Rosen (EPR) pair is essentially a unique kind of entanglement; i.e., we can evaluate any bipartite entangled state by the number of EPR pairs. The situation is totally different in the multipartite setting. We readily see that two EPR pairs, distributed over 3 parties, can create the so-called Greenberger-Horne-Zeilinger (GHZ) 3-qubit entanged state and in turn the GHZ state can create one EPR pair under local operations and classical communication (LOCC). But the opposite conversions are never possible under LOCC. This may imply that the GHZ state is "irreducible" to EPR pairs. In general, we would ask, what kinds of essentially different multipartite entanglement there exist in this situation (4 qubits distributed over *3 parties*)? Observing that the above example can be seen as multiparties LOCC protocols such as the creation of the GHZ state and entanglement swapping, our work turns out to be also useful for their better understanding.

In this talk, (i) we give the complete classification of multipartite entangled states in the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$ under SLOCC. Our study can be seen as the first example of the SLOCC classification of multipartite entanglement where one of multiparties has more than one qubits. We show that nine classes constitute the five-graded partially ordered structure of Fig. 2. Remarkably, a unique maximally entangled class lies on its top, in contrast with the l-qubit $(l \geq 3)$ cases. We also present a convenient criterion to distinguish these classes by SLOCC-invariant entanglement measures.

(ii) We illustrate that important LOCC protocols in QIP are given as noninvertible (downward) flows between different entangled classes in the partially ordered structure of Fig. 2. In particular, we show that two EPR pairs are powerful enough to create any state with certainty in our situation.

Let us begin with the setting. We here address the SLOCC classification of entanglement, which is a *coarse-grained* classification under LOCC. Consider the single copy of a multipartite pure state (precisely, a ray) $|\Psi\rangle$ on the Hilbert space $\mathcal{H} = \mathbb{C}^{k_1} \otimes \cdots \otimes \mathbb{C}^{k_l}$,

$$|\Psi\rangle = \sum_{i_1,\ldots,i_l=0}^{k_1-1,\ldots,k_l-1} \psi_{i_1\ldots i_l} |i_1\rangle \otimes \cdots \otimes |i_l\rangle.$$

In LOCC, we recognize two states $|\Psi\rangle$ and $|\Psi'\rangle$ which are interconvertible deterministically, e.g., by local unitary operations, as the equivalent entangled states. On the other hand in SLOCC, we identify two states $|\Psi\rangle$ and $|\Psi'\rangle$ which are interconvertible probabilistically, i.e., with a nonvanishing probability, since they are supposed to perform same tasks in QIP in different success probabilities. Mathematically, $|\Psi\rangle$ and $|\Psi'\rangle$ belong to the same SLOCC entangled class if and only if they can convert to each other by *invertible* SLOCC operations,

$$|\Psi'\rangle = M_1 \otimes \cdots \otimes M_l |\Psi\rangle,$$

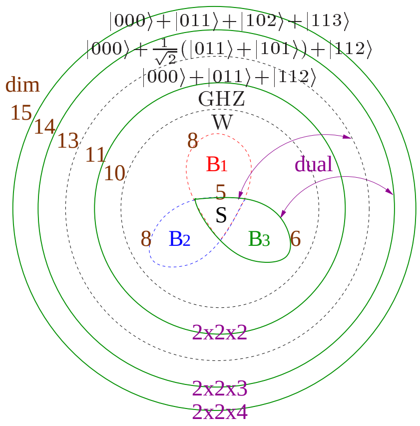

FIG. 1: The onion-like classification of multipartite entangled classes (SLOCC orbits) in $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$ $(n \geq 4)$. There are nine classes divided by "onion skins" (the orbit closures). In the "bipartite" (AB)-C picture, these classes merge into four classes, divided by the skins of the green solid line, so that we can perform LOCC operations more freely.

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*Electronic address: `miyake@monet.phys.s.u-tokyo.ac.jp`
†Electronic address: `frank.verstraete@mpq.mpg.de`

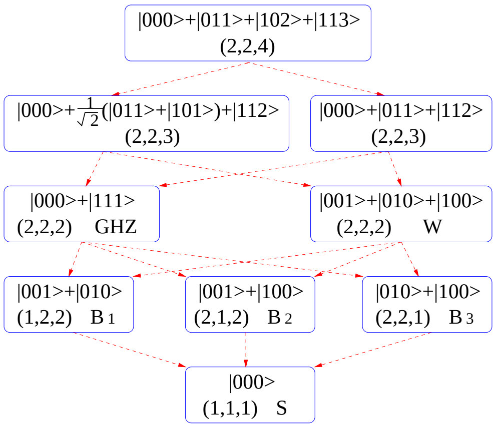

FIG. 2: The five-graded partially ordered structure of nine entangled classes in the $2 \times 2 \times n$ $(n \geq 4)$ case. Every class is labeled by its representative, its set of local ranks, and its name. Noninvertible SLOCC operations, indicated by dashed arrows, degrade higher entangled classes into lower ones.

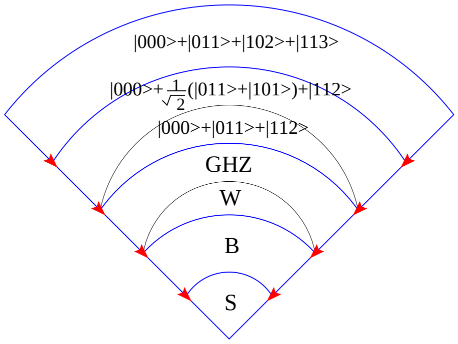

FIG. 3: The SLOCC classification of multipartite mixed states in the $2 \times 2 \times n$ $(n \geq 4)$ cases. Each mixed state ρ in the class labeled by $|\Psi\rangle$ is convex combinations of pure states *inside* the "onion skin" of $|\Psi\rangle$ in Fig. 1. So the outer ρ is, the more kinds of entangled pure states it contains.

where M_i is any local operation having a nonzero determinant on the i-th party, i.e., $M_i \in GL(k_i, \mathbb{C})$. It can be also said that an invertible SLOCC operation is a completely positive map followed by the postselection of a successful outcome. After distinguishing inequivalent SLOCC entangled classes, we would characterize their relationship (the structure of entanglement) under *noninvertible* SLOCC local operations.

Theorem 1 *Consider pure states in the Hilbert space* $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$ $(n \geq 4)$, *they are divided into nine entangled classes, seen in Fig. 1, under invertible SLOCC operations. These nine entangled classes constitute five-graded partially ordered structure of Fig. 2, where noninvertible SLOCC operations degrade higher entangled classes into lower entangled ones.*

The theorem gives the complete classification of multipartite pure entangled states in $2 \times 2 \times n$ $(n \geq 4)$ cases. It naturally contains the classification for the $2 \times 2 \times 2$ (3-qubit) case and the $2 \times 2 \times 3$ case. We find that SLOCC orbits are added outside the onion-like picture (Fig. 1) and the partially ordered structure (Fig. 2) becomes higher, as the third party Clare has her larger subsystem. Remarkably, for the $2 \times 2 \times n$ $(n \geq 4)$ cases, the generic class is only one "maximally entangled" class located on the top of the hierarchy. This is a clear contrast with the situation of the $2 \times 2 \times 2$ and $2 \times 2 \times 3$ cases, where there are two different entangled classes on its top. It suggests that, even in the multipartite situation, there is a unique entangled class which can serve as resources to create any entangled state, if the Clare's subsystem is large enough. Our result not only describes the situation that only Clare has the abundant resources, but also would be useful in analyzing entanglement of 2-qubit mixed states.

The onion-like SLOCC classification of pure states can be extended to mixed states. A multipartite mixed state ρ can be written as a convex combination of projectors onto pure states (extremal points), $\rho = \sum_i p_i |\Psi_i(\mathcal{O})\rangle\langle\Psi_i(\mathcal{O})|, \quad p_i > 0$, where $|\Psi_i(\mathcal{O})\rangle$ is the pure state of the SLOCC entangled class $\mathcal{O}$. Our idea is to discuss how ρ needs at least the outer entangled class $\mathcal{O}$ of the onion picture Fig. 1. That is how we classify mixed states in a SLOCC-invariant, totally ordered way (of Fig. 3), depending on the different kinds of resources (multipartite pure entangled states) they contain.

Now, we briefly discuss two characteristics of multipartite entanglement in our situation. (i) Observe that two EPR pairs, initially prepared over 3 parties in the entanglement swapping, are equivalent to the representative of the generic entangled class,

$$\begin{aligned}|2\text{ EPR}\rangle &= (|00\rangle + |11\rangle)_{AC_1} \otimes (|00\rangle + |11\rangle)_{BC_2} \\ &= |00(00)\rangle + |01(01)\rangle + |10(10)\rangle + |11(11)\rangle_{ABC_{12}}.\end{aligned}$$

We can say that the entanglement swapping is a LOCC protocol creating the isolated (maximal) bipartite entanglement between Alice and Bob from generic entanglement. In other words, it is given by a downward (invertible) flow in Fig. 2 form the generic class to the biseparable class B_3. Two EPR pairs also can create two different kinds of the genuine 3-qubit entanglement, GHZ and W. These LOCC protocols are given by downward flows from the generic class to the GHZ and W class, respectively. That is how we see that important multi-party LOCC protocols in QIP are given as noninvertible (downward) flows in the partially ordered structure, such as Fig. 2, of multipartite entangled classes. So, the SLOCC classification of multipartite entanglement can be expected to give us an insight in looking for new novel LOCC protocols.

(ii) We can show that two EPR pairs are powerful enough to create any state *with certainty* in our $2 \times 2 \times n$ cases. Generally, we find that when one of multiparties holds at least a half of the total Hilbert space, the situation is somehow analogous to the bipartite cases. The maximally entangled state, i.e., the representative of the generic class, can create any state with certainty.

Reference:
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