Quantum Logic for Quantum Programs

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Abstract. We present a way to use quantum logic for the study of quantum programs. This is made possible by using an extension of the usual propositional language in order to make transformations performed on the system appear explicitly. This way, the evolution of the system becomes part of the logical study. We show how both unitary operations and two-valued measurements can be included in this formalism and can thus be handled logically.

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The logical study of quantum mechanics, originated in the thirties by von Neumann and Birkhoff [BvN36], aims at investigating formally what makes quantum mechanics so different from the classical world. To quote the pioneering article:

> "*One of the aspects of quantum theory which has attracted the most general attention, is the novelty of the logical notions which it presupposes... The object of the present paper is to discover what logical structures one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic.*"

The starting point of this study is based on the use of *closed subspaces* of a Hilbert space $\mathcal{H}$ to represent properties about the system. Operations on subspaces, such as orthocomplementation and intersection, are interpretations of the negation and conjunction of logical propositions, thus allowing to define a full-fledged propositional logic. This constitutes the *standard quantum logic* or *orthomodular quantum logic* [Hug89, Svo98, DCG01].

Since its origins, many variations have been studied, and different attempts have been made to identify some axioms or conditions which would permit to recapture the Hilbert space formalism [Mac57, Pir76, PP91]. Unfortunately, despite the large amount of publications on this topic, these works have remained extremely theoretical, and have led to very little applications. However, it is possible to use the quantum logic formalism to express and study properties in a quantum computation context, by extending the language in order to have quantum operations appear explicitly and thus having the possibility to include the evolution of a system in the logical study.

In the present article, we present such a kind of extension of the quantum logic formalism. It is based on the use of closed subspaces as partial descriptions of states of the system (with statements of the form "the actual state lies in this subspace"). This is slightly different from the usual approach, since a notion of *approximation* is present, and logical assertions can then be seen as relating knowledge about the system's state at different moments during the computation. From this starting point, we introduce a collection of unary operators which do all correspond to an action performed on the system and represent knowledge about the system after this action has been performed. Two kinds of action are presented: the application of an unitary operator, and the performance of a measurement.

We show that the application of unitary operators on a quantum system is information preserving, which is formalized logically by an equivalence. On the contrary, the performance of a measurement, which can be easily formalized logically by a disjunction, generally leads to a loss of information.

Finally, we illustrate this logical formalism by applying it to the study of a simple qubit teleportation circuit [BBC98].

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