Generating optical nonlinearity using trapped atoms

W. J. Munro[1] Alexei Gilchrist[2] G. J. Milburn[2] Kae Nemoto[3]

[1] *Hewlett-Packard Laboratories, Filton Road, Stoke Gifford, Bristol BS34 8QZ, United Kingdom*
[2] *Centre for Quantum Computer Technology, The University of Queensland, St Lucia, QLD, 4072 Australia..*
[3] *National Institute of Informatics, 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430*

Abstract. We present a scheme for producing an optical nonlinearity using an interaction with one or more ancilla two-level atomic systems. The nonlinearity together with general linear transformations is sufficient for simulating arbitrary Hamiltonian evolution on a Fock state qudit. We give two examples of the application of this nonlinearity: one for the creation of nonlinear phase shifts on optical fields, and the other for the preparation of optical Schrödinger cat states.

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1 Introduction

Our ability to perform quantum transformations on optical fields is hampered by the lack of materials with intrinsic optical non-linearities. While it is possible to circumvent this problem with schemes conditioned on photo-detection (see for instance [1, 2]), efficiency problems are encountered both due to the inherent nature of the schemes and efficiencies of current photo-detectors. In this paper we propose the use of the interaction with a simple atomic system, conditioned on high efficiency atomic measurements, to generate a near-deterministic (with probability approaching one) nonlinearity on an optical field. This nonlinearity, together with linear transformations, is sufficient for generating arbitrary Hamiltonian evolution on a qudit formed from a truncated sequence of Fock states.

The interaction between the single mode field and a two-level atom is described by the effective Hamiltonian $\boldsymbol{H} = \kappa(\boldsymbol{a}^\dagger\boldsymbol{\sigma}^- + \boldsymbol{a}\boldsymbol{\sigma}^+)$ where $\boldsymbol{a}^\dagger$, $\boldsymbol{a}$ are the boson creation and annihilation operators for the field mode and the operators $\boldsymbol{\sigma}^+ = |e\rangle\langle g|$, $\boldsymbol{\sigma}^- = |g\rangle\langle e|$ are the raising and lowering operators for the atomic state. The unitary transformation that acts when this interaction is applied for a time t is $\boldsymbol{U}(\tau) = \exp[-i\tau(\boldsymbol{a}^\dagger\boldsymbol{\sigma}^- + \boldsymbol{a}\boldsymbol{\sigma}^+)]$ where $\tau = \kappa t$. If the atom is prepared in the ground state and found in the ground state after the interaction, the conditional state of the field is given by

$$\boldsymbol{\Upsilon}_g(\tau)|\psi\rangle = \cos(\tau\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}})|\psi\rangle \tag{1}$$

2 The $\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}}$ operator

It is first instructive to consider a unitary operator of the form $\exp(i\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}})$. A nonlinearity of the form $\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}}$, which we refer to as the square-root-number operator, turns out to be as good as a Kerr nonlinearity for universal quantum computation in the infinite dimensional Hilbert space of a Harmonic oscillator [3]. The concept of universal computation is isomorphic to the concept of universal simulation, which is the ability to simulate any arbitrary Hamiltonian evolution, to any degree of accuracy, by combining the evolution due to a fixed class of Hamiltonian generators.

To motivate this idea, consider what kind of nonlinear oscillator would correspond to the Hamiltonian $\boldsymbol{H}_{\rm srn} = \sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}}$. The resulting Heisenberg equation of motion for the field amplitude operator is

$$\dot{\boldsymbol{a}} = i[\boldsymbol{H}_{\rm srn}, \boldsymbol{a}] = -i(\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}+1} - \sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}})\boldsymbol{a} \tag{2}$$

which clearly indicates an oscillator in which the frequency is state dependent. The right hand side may be expanded as a power series in $(\boldsymbol{a}^\dagger\boldsymbol{a})^{-1/2}$ to give

$$\dot{\boldsymbol{a}} = -i\left(\frac{1}{2}(\boldsymbol{a}^\dagger\boldsymbol{a})^{-1/2} - \frac{1}{8}(\boldsymbol{a}^\dagger\boldsymbol{a})^{-3/2} + \dots\right)\boldsymbol{a} \tag{3}$$

This corresponds to an oscillator for which the frequency of oscillation *decreases* with increasing energy. For comparison, the Kerr nonlinear oscillator corresponds to an oscillator in which the frequency increases with increasing energy. It is thus clear that the square-root-number Hamiltonian will result in a rotational shearing of states in the phase plane of the oscillator in much the same way as occurs for the Kerr nonlinearity [4].

We can go further by using the results of Braunstein and Lloyd [3] who considered the question of what Hamiltonians are universal for quantum computation in an infinite dimensional Hilbert space. Their results show that Hamiltonians that are at most quadratic in the canonical momentum and position variables are not universal. For instance, in the case of a single mode field with annihilation and creation operators $\boldsymbol{a}, \boldsymbol{a}^\dagger$, successive applications of Hamiltonians from the set of displacements, squeezing and rotations, $\mathcal{H}_{\rm lin} = \{z\boldsymbol{a} + z^*\boldsymbol{a}^\dagger, z\boldsymbol{a}^2 + z^*(\boldsymbol{a}^\dagger)^2, z\boldsymbol{a}^\dagger\boldsymbol{a}\}$, can generate arbitrary linear canonical transformations in the phase space variables but no other transformations. A universal set is easily obtained by adjoining almost any Hamiltonian that is at least cubic in the canonical variables. A universal set of Hamiltonians could be made up from $\mathcal{H}_{\rm lin}$ together with the Kerr nonlinearity $\boldsymbol{H}_k = (\boldsymbol{a}^\dagger)^2\boldsymbol{a}^2$. Another choice is the cubic Hamiltonian $\boldsymbol{H}_c = \boldsymbol{a}^\dagger\boldsymbol{a}(\boldsymbol{a}+\boldsymbol{a}^\dagger) + hc$. We can now show that the square-root-number operator $\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}}$ together with the set $\mathcal{H}_{\rm lin}$ can be used to simulate a cubic Hamiltonian, and thus is universal for quantum simulations in the Hilbert space of a single mode.

Consider acting on the unitary operator generated by the square-root-number operator with a large displacement α: $\boldsymbol{U}(\alpha,\theta) = \boldsymbol{D}^\dagger(\alpha)\exp(i\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}})\boldsymbol{D}(\alpha)$ where $\boldsymbol{D}(\alpha)$ is the displacement operator given by $\boldsymbol{D}(\alpha) =$

$\exp(\alpha \boldsymbol{a}^\dagger - \alpha^* \boldsymbol{a})$. The resulting unitary operator may then be written as

$$\boldsymbol{U}(\alpha,\theta) = e^{i\theta|\alpha|}\exp\left[i\theta\left(\frac{\boldsymbol{x}(\phi)}{2}+\frac{\boldsymbol{a}^\dagger\boldsymbol{a}}{2|\alpha|}-\frac{\boldsymbol{x}(\phi)^2}{8|\alpha|} -\frac{\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{x}(\phi)+\boldsymbol{x}(\phi)\boldsymbol{a}^\dagger\boldsymbol{a}}{8|\alpha|^2}+O(|\alpha|^{-3})\right)\right] \quad (4)$$

where $\boldsymbol{x}(\phi) = (\boldsymbol{a}e^{-i\phi} + hc)$ with ϕ the phase of α. The first three terms in the argument of the exponential correspond to a second order Hamiltonian and simulate a displacement, a rotation, and a squeezing operation respectively. The fourth term however is a cubic term which is what we required for a universal set of Hamiltonians for a single mode. By using a linear Hamiltonian to mix several modes (for instance a beam-splitter), arbitrary multi-mode Hamiltonians can be constructed [3]. It is thus clear that we can use the square-root-number operator, together with an arbitrary linear transformations to perform universal computation.

3 Universality of $\cos\left[\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}}\right]$?

Now that we have shown the universality of the operator $\exp(i\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}})$ what can we say directly about $\cos\left[\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}}\right]$? The action of this conditional operator on the number state $|n\rangle$ is to multiply the state by the amplitude $\cos(\theta\sqrt{n}) = [\exp(i\theta\sqrt{n}) + \exp(-i\theta\sqrt{n})]/2$. Clearly, if we can restrict the interaction so that this amplitude is ±1, then it is equivalent to the full unitary operator— and we might expect to exploit the nonlinear interaction in a similar way. Now, as it turns out, it *is* possible to choose θ such that for a finite size computational space, $|0\rangle \ldots |N\rangle$,

$$\cos\left[\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}}\right]|n\rangle \approx \begin{cases} -|n\rangle & \text{for } n = 2\,(2m+1)^2 \\ |n\rangle & \text{for } n \neq 2\,(2m+1)^2 \end{cases} \quad (5)$$

We have chosen the first sign shift to occur on the $|2\rangle$ Fock state. This is an arbitrary choice and other terms can be shifted instead. With this nonlinear Hamiltonian, it is possible in conjunction linear Hamiltonians to generate arbitrary multi-mode Hamiltonians. Hence we can in principle create any of the unitary operators required for universal qutrit computation. This argument extends to higher qudits in a direct manner.

4 Applications

One natural application of this result is for the creation of Knill, Laflamme and Milburn (KLM) [1] nonlinear sign shift (NS) gate. This gate produces the phase shift defined by

$$c_0|0\rangle + c_1|1\rangle + c_2|2\rangle \rightarrow c_0|0\rangle + c_1|1\rangle - c_2|2\rangle \quad (6)$$

If the ancilla modes in the KLM configuration are replaced with a two level atom similar conditional nonlinear phase shifts can be achieved by near deterministicnpost-selection on atomic measurements. It is straightforward to show that the operator $\boldsymbol{\Upsilon}_g$ acting on the general two photon state will leave one with the state $A_0c_0|0\rangle + A_1c_1|1\rangle + A_2c_2|2\rangle$ where $A_0 = 1$, $A_1 = \cos(\tau)$ and $A_2 = \cos(\sqrt{2}\tau)$. Solutions for which (A_1, A_2) approaches arbitrarily close to $(1,-1)$, thus near deterministically implementing the NS gate can easily found, for instance, for $\tau = 219.918$, $A_1 = -A_2 = 0.999979$.

Lastly let us consider the case where the field in an initial coherent state $|\alpha\rangle$. It is easily shown that the conditional transformation, $\cos(\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}})$, generates a coherent superposition of coherent states, a so called *Schrödinger cat state*. In figure [1] we plot the Q function $Q(\beta) = |\langle\beta|\cos(\theta\sqrt{\boldsymbol{a}^\dagger\boldsymbol{a}})|\alpha\rangle|^2$ as a function of the two canonical phase space variables x and p for several choices of α and θ. We observe that if we load a coherent state into the cavity, and repeat the conditioning measurement procedure outlined previously, the conditional state of the field will be prepared in a state which is close to a cat state.

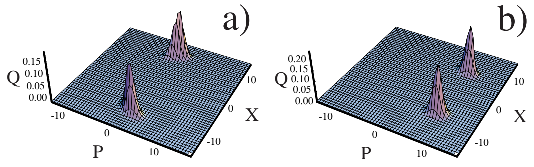

Figure 1: A plot of the Q-function versus the two canonical phase space variables, for conditional state produced from an initial coherent state using, (a) $\alpha = 10,\ \theta = 10\pi$ and (b) $\alpha = 10,\ \theta = 5\pi$

5 Discussion

We have shown how the ability to do very efficient measurements on single atoms trapped in an optical cavity can be used to implement nonlinear conditional phase shifts on the intra-cavity field. By carefully choosing the interaction time, a nonlinear interaction can be implemented with near unit probability, that, together with linear transformations, is universal for simulating an interaction on qudits. For small qudits the required interaction time can easily be found numerically. If the field state can be carefully switched in and out of the cavity, the method can be used to implement near-deterministic nonlinear gates for quantum optical computing. It can also be used to conditionally generate coherent superpositions of coherent states and thus can provide the key resource for quantum computing with coherent states [5].

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