Two-qubit module design of a universal quantum compiler

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We present a theoretical framework and tools to design two-qubit modules of a universal quantum compiler. This compiler constitutes an interface connecting a theoretical quantum simulation or computation algorithm, given in terms of elementary gates, with its physical implementation.

We particularly focus on two-qubit operations which typically represent a crucial and more challenging component of physical implementations of quantum information processing compared to single qubit gates. In the framework of quantum mechanics, the two-qubit unitary operations are represented by the Lie group $SU(4)$. This group factorizes into the subgroup of local gates $SU(2) \otimes SU(2)$ and the set of non-local operations $SU(4) \backslash SU(2) \otimes SU(2)$. Entangling unitary operations are all non-local gates except the SWAP gate.

Our particular objective is a universal quantum circuit which simulates any arbitrary two-qubit operation given either any physical interaction between qubits or any entangling unitary operation. We assume that these interactions or operations can be combined with any local gates in an arbitrary circuit.

1 Two-qubit local equivalence classes

Our theoretical framework is the geometric theory of two-qubit operations [1] constructed using the Cartan decomposition of the Lie algebra $\mathfrak{su}(4)$ [2], two-qubit local invariance theory [3], and the Weyl group theory [2].

1.1 Cartan decomposition

Two unitary transformations U_1, $U_2 \in SU(4)$ are called *locally equivalent* if they differ only by local operations: $U_1 = k_1 U_2 k_2$, where k_1, $k_2 \in SU(2) \otimes SU(2)$. The set of all operations locally equivalent to some U is called the local equivalence class of U, and is denoted $[U]$. From the Cartan decomposition of $\mathfrak{su}(4)$ [1], any two-qubit gate $U \in SU(4)$ can be written in the following canonical form

$$U = k_1 A k_2 = k_1 \exp\{\frac{i}{2}(c_1 \sigma_x^1 \sigma_x^2 + c_2 \sigma_y^1 \sigma_y^2 + c_3 \sigma_z^1 \sigma_z^2)\} k_2. \tag{1}$$

The non-local component A of U is generated by the maximal Abelian subalgebra $\mathfrak{a}$ of the $\mathfrak{su}(4)$ algebra [2]. Because the two-qubit gate U is periodic in c_k (with the period π), its local equivalence class can be represented as a point on a three torus $T^3 = S^1 \times S^1 \times S^1$, or alternatively as a point $[c_1, c_2, c_3]$ in a three dimensional cube $C \in \mathbb{R}^3$, since $T^3 \cong \mathbb{R}^3/\mathbb{Z}^3$. The Cartan decomposition thus provides an intuitive three dimensional representation of unitary operations focusing on their entangling content via local equivalence classes and disregarding unimportant local gates k_1 and k_2.

1.2 Local invariants.

The representation given by the Cartan decomposition does not uniquely characterize the local equivalence classes because different points in the cube C may correspond to the same class.

The local invariant theory is employed to resolve this problem. We connect the local invariants of Makhlin [3], which uniquely characterize local equivalence classes, to the coordinates $[c_1, c_2, c_3]$. The details of the construction can be found in [1]. Using the Cartan decomposition we can express the local invariants of U as:

$$\begin{aligned} G_1 &= \cos c_1 \cos c_2 \cos c_3 \\ G_2 &= \sin c_1 \sin c_2 \sin c_3 \\ G_3 &= 2(\cos^2 c_1 + \cos^2 c_2 + \cos^2 c_3) - 3 \end{aligned} \tag{2}$$

Note that in this representation, any permutation of the coefficients $[c_1, c_2, c_3]$ with or without sign flips leaves the local invariants G_1, G_2 and G_3 unchanged. This indicates an extra symmetry in the cube C which is removed using Weyl group theory [2].

The final representation of the local equivalence classes is given in a form of a geometric three-dimensional object (a tetrahedron) called Weyl chamber $\mathfrak{a}^+$. Each point inside this object is in one-to-one correspondence with one local equivalence class.

2 Quantum evolution

The quantum evolution under a physical Hamiltonian – generally including both local and non-local (interaction) terms – is represented as a one-dimensional subset, a *trajectory* in the Weyl chamber. This provides an intuitive perspective on unitary operations generated by a given dynamics. While the non-local terms connect different local equivalence classes (points in the chamber)

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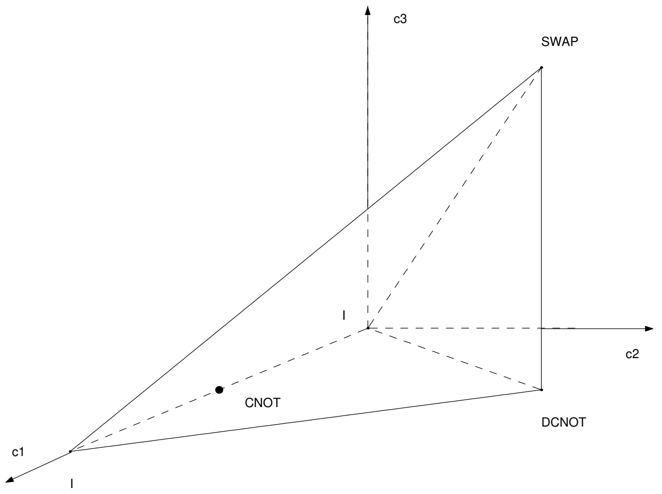

Figure 1: The Weyl chamber provides intuitive three-dimensional representation of the two-qubit unitary operations in terms of the local equivalence classes. Indicated are the local equivalence classes of identity (I) (periodic along the c_1 axis), controlled-NOT (CNOT) gate, SWAP gate and double-CNOT (DCNOT) operation.

by changing the entangling content of generated operations with time, the local terms are responsible for the direction of this evolution.

We provide analytical expressions for the Weyl chamber trajectories for several exemplary cases important in quantum computation with solid state and superconducting qubits. Both general and specific results, applicable to the optimal control of qubit systems, are important in the design of universal quantum circuits (UQC).

3 UQC from a physical interaction

The distinct role of local and non-local Hamiltonian terms in quantum evolution suggests that combination of these terms in a short quantum circuit may result in generation of any unitary operation from SU(4).

We develop a systematic way to construct quantum circuits from a given physical Hamiltonian that can simulate any arbitrary two-qubit gate exactly. We consider here the conventional scenario of a Hamiltonian acting on a layout of qubits such that any arbitrary single qubit operation and certain specific two-qubit operations may be turned on for selected time durations and applied in series. We explicitly construct a quantum circuit that contains three non-local gates generated by a given two-body interaction Hamiltonian for corresponding finite time durations, together with at most four local gates. The circuit structure based on at most three applications of a non-local interaction is fundamentally linked to the three-dimensional geometric representation of two-qubit unitary operations.

4 UQC from a unitary operation

We also provide an analytic way to implement any arbitrary two-qubit unitary operation, given an entangling two-qubit gate together with local gates [4]. This provides explicit construction of a universal quantum circuit that exactly simulates arbitrary two-qubit operations. Each block in this circuit is given in a closed form solution. We also provide a uniform upper bound on the number of applications of the given entangling gates, and find that exactly half of all the Controlled-Unitary gates satisfy the same upper bound as the controlled-NOT gate.

5 UQC from CNOT

We improve the general uniform upper bounds on universal quantum circuit, obtained for the input of an arbitrary non-local operation, by restricting this input to the CNOT gate. Our proof, which is an alternative to the proof provided recently by Vidal and Dawson [5, 6], requires at most three CNOT gates to construct any $U \in SU(4)$. In addition, we atempt to extend our results beyond the CNOT gate.

6 Conclusion

We presented a theoretical framework and analytic tools to design two-qubit modules of a universal quantum compiler. We briefly introduced an intuitive geometric picture of the group of two-qubit operations, obtained by tranformation of the Lie group of two-qubit operations to intuitive representation in terms of local equivalence classes, a Weyl chamber: $SU(4) \to T^3 \cong \mathbb{R}^3/\mathbb{Z}^3 \to \mathfrak{a}^+$, and provided its main applications. These include (i) generation of non-local gates from physical interactions, (ii) design of universal quantum circuits, (iii) analytical construction of exact quantum circuit, and (iv) formulation of uniform upper bounds for this construction. Our results are useful in quantum computation, quantum simulation and quantum control.

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