Accessibility of physical states and non-uniqueness of entanglement measure

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Abstract. Ordering physical states is the key to quantifying some physical property of those states uniquely. Bipartite pure entangled states are totally ordered under LOCC in the asymptotic limit and uniquely quantified by the well-known entropy of entanglement. However, we show that mixed entangled states are partially ordered under LOCC even in the asymptotic limit. Therefore, non-uniqueness of entanglement measure is proven based on an *operational* notion of asymptotic convertibility.

Keywords: Entanglement measure, Partially ordered set, Thermodynamics, Non-uniqueness

Accessibility between two physical states by some physical process is crucial in being able to compare the states quantitatively. When there exists an *operation* that converts one state to another, we can derive an ordering between the two states from the accessibility based on this operation. This ordering (together with a few other natural assumptions) makes it possible to define a quantity that compares the states. However, if it is impossible to convert one state into another in either direction within a given framework, there exists no coherent way to compare those two states.

Uniqueness of a measure that quantifies a certain physical property is strongly related to ordering of states. When all elements in a given set of physical states can be completely ordered, i.e., arbitrary two states can be ordered (total order), we can make at least one consistent measure that quantifies the set. However, if there exists no ordering that works globally, i.e., a certain pair of states cannot be ordered (partial order), then we fail to find a consistent way to "align" all the states. In other words, total order is a necessary condition for a set to be quantified by the unique measure.

One of the most familiar examples in physics that contain partial order is in special theory of relativity. A pair of events in the space-time that include each other in their light-cone (i.e., the interval between the two events is time-like) are accessible because one can affect the other by sending some signal. However, if one is outside of the light-cone of the other (i.e., the interval between the two events is space-like), then it is impossible to connect them by any physical operation. There exists no unique way of ordering such two states; different orderings are possible by choosing different reference frames. Therefore, the set of events is a partially ordered one, which leads to the well-known non-uniqueness of simultaneity that follows from the principles of special theory of relativity (See Chapter 17 of [1], for example). Furthermore, in a modern approach to relativity, a fundamental structure of spacetime is modeled as a partially ordered set called a causal set [2].

The most beautiful and successful application of the theory of ordering physical states is in thermodynamics, where all equilibrium states are totally ordered under adiabatic processes and quantified by the unique measure of entropy S. Given two equilibrium states (A and B), A can access B via an adiabatic process iff $S(A) \leq S(B)$. (If the equality holds, B can also access A and so the process becomes reversible.) The uniqueness of entropy was proven by Giles with his axiomatic approach, which was developed to clarify the structure of thermodynamics [3]. (This approach has recently been revisited in Ref. [4].)

It has been shown recently that thermodynamics and theory of quantum entanglement share the same mathematical structure. Adiabatic processes in thermodynamics correspond to manipulations of bipartite entangled pure states by local operations and classical communication(LOCC) [5]. Therefore, bipartite entangled pure states are totally ordered under LOCC in the asymptotic limit, and entropy gives the unique measure in the context as well (known as the von Neumann entropy of entanglement [6]).

Contrary to the case of bipartite pure states, the unique measure of entanglement in mixed states has not been established yet. It has been proven that if two entanglement measures coincide in pure states but differ in mixed states, then they impose different orderings [7]. In fact, some entanglement measures proposed so far are different.

In this work, we rigorously prove that there exists no unique way of defining entanglement measure in mixed states by proving that bipartite mixed states are partially ordered under LOCC [8]. In other words, we completely refute the possibility of the unique measure by showing that different orderings are inevitable under LOCC. The proof invokes Giles's axioms, especially Axiom 5, which reads *If two states A and B are both accessible from another state C, then A and B are accessible in either direction (or both).* This is the exactly what distinguishes total order from partial one. We show that entangled mixed states are partially ordered by giving a counterexample to Axiom 5. (For other natural axioms and details

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of Giles's approach, see Refs. [3, 5].)

The rigorous definition of accessibility here is as follows: A state ρ is convertible into a state σ if and only if for every (arbitrarily small) real number ϵ, there exists an integer n_0, and a sequence of LOCC L_n such that for any integer $n \geq n_0$ we have that $\| L_n(\rho^{\otimes n}) - \sigma^{\otimes n} \| \leq \epsilon$, where $\rho^{\otimes n} = \rho \otimes \rho \cdots \otimes \rho$ represents a tensor product of n copies of the state ρ and $|| \cdots ||$ denotes the usual trace norm distance between two mixed quantum states. With this definition, we discuss convertibility between two different states of the *same* number of copies.

Intuitively, bipartite mixed states that are most likely to fail this axiom are bound entangled states [9]. Since bound entangled states are mixed states from which no entangled pure state can be distilled, if we take one of those and a pure entangled state as a pair of possible candidates for a counterexample, the first half of the proof has already been accomplished by definition. So, all we have to do is to prove the inconvertibility in the opposite direction.

In order to prove that, we take a certain positive-partial-transposition (PPT) bound entangled state $\rho_{\rm AB}$ of a 3×3 system given in Ref. [10]. The important fact about the state $\rho_{\rm AB}$ is that its entanglement cost $E_C(\rho_{\rm AB})$ is positive, which is defined as $E_C(\rho) \equiv \lim_{n\to\infty} E_f(\rho^{\otimes n})/n$ [11], where $E_f(\rho)$ represents the entanglement of formation of ρ [12]. Owing to this property, one can choose an entangled pure state $\sigma_{\rm AB} = |\phi\rangle\langle\phi|$ such that $0 < E_C(\sigma_{\rm AB}) < E_C(\rho_{\rm AB})$. For simplicity, we choose $|\phi\rangle$ to be an entangled states with Schmidt number two or three, i.e., a 2×2 or 3×3 system. Since the entanglement cost E_C is an entanglement monotone, i.e., it cannot increase under LOCC, $\sigma_{\rm AB}$ can never be converted into $\rho_{\rm AB}$ even asymptotically. The monotonicity of entanglement cost E_C can easily be derived from the fact that entanglement of formation E_F is also an entanglement monotone. Note that the above inconvertibility holds in the sense of the *same* number of copies. Otherwise, a sufficiently large number of copies of $\sigma_{\rm AB}$ can always produce a much smaller number of copies of $\rho_{\rm AB}$ with certainty.

Besides the above fact, note that a maximally entangled state $|\Phi_3\rangle_{\rm AB}$ with Schmidt number three can access both $\rho_{\rm AB}$ and $\sigma_{\rm AB}$ without reducing the number of copies. Therefore, we found a counterexample that two states $\rho_{\rm AB}$ and $\sigma_{\rm AB}$ are *not* convertible into each other in spite of the fact that both of them can be accessed from the common state $|\Phi_3\rangle_{\rm AB}$. As we mentioned above, violation of Axiom 5 means that entangled mixed states are partially ordered under LOCC in the asymptotic limit. (We note that though, in Giles's axioms, transformations assisted by asymptotically negligible amount of auxiliary states are considered, the undistillable property of bound entanglement remains unchanged even with an assistance of auxiliary entangled states [10].)

In the above argument, we chose a pure state as $\sigma_{\rm AB}$ for simplicity. However, it can be replaced with distillable mixed states $\chi_{\rm AB}$ such that $0 < E_C(\chi_{\rm AB}) < E_C(\rho_{\rm AB})$ because PPT bound entangled states cannot be converted into negative-partial-transposition (NPT) states by LOCC. Thus, the above also holds for any such $\chi_{\rm AB}$. Generally, it can be concluded that any PPT bound entangled state with positive entanglement cost always has an NPT state that is not convertible into each other.

Non-uniqueness of entanglement measure in mixed states immediately follows from the fact that the set of mixed states is a partially ordered one under LOCC in the asymptotic limit. Since there is no *operational* way to link an inconvertible pair of states, there seems to be no way of assigning "meaningful" amounts of entanglement to them that could determine which state is more entangled. In other words, we have therefore proven a sort of "relativity" of entanglement measure under asymptotic convertibility with LOCC, which means that there exists no *absolute* entanglement measure at least under LOCC. An important future direction is finding out exactly how the thermodynamical structure breaks down when mixedness appears in entanglement.

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