Jump Codes and Other Quantum Codes

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Jump codes were developed for purpose of stabilization of errors in quantum computations in the context of quantum optics. We explore the connections between Jump codes, the Pauli error group and topological quantum codes. We show that several previously constructed jump codes are instances of a generalised construction.

1 Introduction

In this paper we make some further observations about a new type of quantum error correcting codes called *Jump codes*; see [1] and [2]. The initial research on Jump codes concentrated on their constructions and proving various bounds. Several ad hoc constructions of Jump codes were obtained in a combinatorial setting; see [2]. The physical applications of these codes were considered in [1] and in other publications. So far no single construction has emerged for these codes, such as the group theoretic setting of Calderbank, Rains, Shor and Sloane [5]. Nor has the relation between Jump codes and other quantum error correcting codes been satisfactorily explained.

Motivated by these questions we state Theorem 1, which accounts for several of the known Jump codes and is perhaps the natural group theoretic setting for these codes. We note that the Pauli error group is linked with Jump codes in way which is suggestive of the construction of Calderbank et al [5]. There are also hints of a topological connection between Jump codes and Kitaev's [8] construction.

2 Jump Operators

The error correction we consider was motivated by the physical model of a quantum computer in which N two-level atoms (qubits) interact with external laser pulses which synthesize the quantum gates underlying a quantum algorithm. These N qubits are assumed to be arranged in an array (an iontrap) with well defined positions $\mathbf{x}_\alpha$ ($\alpha = 1, \ldots, N$). In addition, these N qubits are assumed to be distinguishable; see [1]. The distinguishability of the qubits guarantees that the dimension of their associated Hilbert space is $K_{\mathcal{H}} = 2^N$ and thus scales exponentially with the number of qubits. In addition, it is assumed that these N qubits couple to the unoccupied modes of the electromagnetic field. This coupling causes spontaneous decay processes of the qubits from their excited states $|1\rangle_\alpha$, to their stable lower lying states $|0\rangle_\alpha$.

Thus in our quantum error correction model the errors are due to spontaneous quantum jumps, i.e., the (excited) state $|1\rangle$ may spontaneously decay into the (ground) state $|0\rangle$. Furthermore, we assume that the decay rate is equal for all subsystems, i.e., independent of the position. Then the corresponding error operator is given by

$$a := |0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

If the operator acts only on the ith subsystem, the following notation will be used:

$$J_i := a_i := \mathrm{id} \otimes \ldots \otimes \mathrm{id} \otimes |0\rangle\langle 1| \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}. \quad (1)$$

The physics of jump codes is described in [1] while the combinatorial approach is developed in [2]. We refer the reader to these sources for an explanation of unfamiliar notions.

3 Constructions

SEED'S were defined in [2]; they are the combinatorial precursors to Jump codes. Jump codes $\mathcal{C} = (v, l, t)_k$ with l states on v qubits are obtained from SEED'S by the encoding

$$|\mathcal{B}^{(i)}\rangle \longrightarrow \frac{1}{\sqrt{|\mathcal{B}^{(i)}|}} \sum_{B \in \mathcal{B}^{(i)}} |B\rangle$$

with $|B\rangle := |\chi_B(i), \ldots, \chi_B(v)\rangle$ (χ_B is the characteristic function of B).

Several of the previously discovered Jump codes are particular cases of Theorem 1 whose proof was given in [4]. To understand this result we need some terminology.

A *block system* $C := (S, \mathcal{B})$ is a finite set $S := \{1, \ldots, n\}$ and a collection of subsets $\mathcal{B} := \{B_1, \ldots, B_k\}$ of S having the same cardinality. The automorphism group of C, $G = \mathrm{Aut}(C)$ consists of all one-to-one mappings of S which preserve the blocks. The *dual* block system C^* has the same underlying set S, but the blocks of C^* are the images under a one-to-one mapping σ of the blocks of $\mathcal{B}$.

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Theorem 1 *Suppose C and C^* are dual block systems of a set $S := \{1, \ldots, n\}$ with a non trivial automorphism group $G = \mathrm{Aut}(C)$. Let G act on C and $\mathcal{O}_{i_j}$ and $\mathcal{O}^{\sigma}_{i_j}$ be disjoint orbits of blocks in C and C^* respectively of cardinality i. Suppose that $t < i$. For each integer s with $1 \le s \le t$ decompose the s-subsets of $\{1, \ldots, n\}$ into orbits $\Theta_s = \{\Theta_{s_1}, \ldots, \Theta_{s_m}\}$ with respect to the induced action of G on the s-subsets. Suppose that for each s the orbits in Θ_s are stabilized by σ, i.e., $\Theta^{\sigma}_{s_i} = \Theta_{s_i}$ for all i. Then the vectors in $\mathcal{O}_{i_j}$ and $\mathcal{O}^{\sigma}_{i_j}$ give a t-SEED$(n, i; 2)$.*

It was shown in [4] that the optimal $(n, \frac{1}{2}\binom{n}{n/2}, 1)_{n/2}$ Jump codes with basis states $1/\sqrt{2}(|x\rangle + |\overline{x}\rangle)$ are an instance of Theorem 1.

4 The Pauli Error Group

We consider the Pauli group G_1 of order 16 on a single qubit. (The n-qubit group G_n is described in [9].) As a matrix group G_1 is generated by the Pauli matrices
It is known that if errors of this type can be corrected on t-qubits, then an arbitrary error on t-qubits, can also be corrected. (See the discussion in [5].)

Besides the two-dimensional unitary representation of G_1 (by the Pauli matrices), G_1 can be also *faithfully* represented as a permutation group of degree eight using the generators: $(1, 2)(3, 6)(4, 7)(5, 8), (1, 3)(2, 5)(4, 8)(6, 7), (1, 4)(5, 6)$. The Pauli group yields a $(8, 2, 2)_3$ Jump code. It has two orbits $\mathcal{O}_0$, $\mathcal{O}_1$ of length 16 on the sets of size 3. These encode the two logical states of $(8, 2, 2)_3$; see [4].

The permutation group G generated by the generators of G_1 and σ has order 32. (σ is the transformation in Theorem 1.) G fuses the orbits $\mathcal{O}_0$ and $\mathcal{O}_1$ into a single orbit $\mathcal{O}$. G has the property that the centre is the same as the derived subgroup $Z(G) = G'$. Also the factor group $G/Z(G)$ is an *elementary abelian* group of order 16 (each non-identity element has order two).

The centre $Z(G)$ is generated by $\tau := (1, 4)(2, 7)(3, 8)(5, 6)$. Since τ stabilizes both of the orbits $\mathcal{O}_0$ and $\mathcal{O}_1$ and $\mathcal{O} = \mathcal{O}_0 \bigcup \mathcal{O}_1$, there is a well defined action of the factor group $G/Z(G)$ on $\mathcal{O}$ which decomposes this orbit into *co-orbits*. Because $G/Z(G)$ is an elementary abelian group, there are 16 co-orbits each having length 2.

This example is suggestive of the group theoretic framework for quantum error correcting codes developed in [5].

5 A Topological Example

In Kitaev's [8] topological approach to quantum error correction the qubits are arranged on the edges of square lattices embedded in a two dimensional torus. Freedman and Meyer considered the non-orientable surface setting and defined quantum error correcting codes by *cellulations* of the projective plane [6]. An example Jump code which is defined by a *triangulation* of a torus.

The triangulation of the torus which we use [3] consists of 14 triangles $\mathcal{T}$ with vertices labeled by the integers in [1..7] and edges labeled by pairs of vertices. The subgroup of the symmetric group S_7, which stablizes $\mathcal{T}$ is a group G_{42} of order 42. G_{42} has a subgroup of index two G_{21} generated by the permutations: $(1, 3, 4, 7, 6, 5, 2), (1, 2, 4)(3, 7, 6)$. G_{21} splits the triangulation $\mathcal{T}$ it into two orbits (orientation classes) of length seven. These orbits encode two logical states of a $(7, 2, 2)_3$ Jump code

To verify that the above encoding gives a $(7, 2, 2)_3$ Jump code, we require a transformation $\sigma := (2, 3)(4, 5)(6, 7)$. Then $\sigma G_{21} = G_{21}\sigma$ and Theorem 1 gives the conclusion.

6 Conclusion

We have shown that several previously defined Jump codes are instances of a general group theoretic setting described in Theorem 1. Some more examples were described in [4]. There are Jump codes where the logical states $|c_0\rangle, |c_1\rangle, \ldots$ are encoded by group orbits of unequal lengths [7]. Seemingly such codes would not fit into the framework of Theorem 1. However, a Jump code with given parameters can be realised with different group actions [4]. For instance the six qubit code $(6, 2, 2)_3$ has two realizations with respect to two distinct groups of order 24. Thus there could be representation of such codes which would fit in with Theorem 1.

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