Self-dual random-plaquette gauge theory and accuracy threshold of the toric code

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Abstract. We study the phase structure of the four-dimensional Z_2 random plaquette gauge theory as a model of topological quantum memory, the toric code in particular. In this model, the procedure of quantum error correction works properly in magnetically ordered phase and phase boundary between ordered and disordered phases gives the accuracy threshold of error correction. Using self-duality of the model, we show that the phase transition property of this model is the same as that of the two-dimensional random bond Ising model, which has been studied very extensively.

Keywords: quantum error correcting, lattice gauge theory, random spin system, self-dual model

The lattice gauge theory was studied mainly from the standpoint of particle physics. However, the lattice gauge theory with randomness has not been studied because of lack of motivation. Recently it is proposed that the random gauge model can be viewed as the model of topological quantum memory [1, 2, 3], where the 2D $\pm J$ random bond Ising model (RBIM) and Z_2 random plaquette gauge model (RPGM) were used. In these models, we suppose that phase error or flip error of qubits occurs due to decoherence and interaction with environment. Fortunately, we can infer the "error chains" of qubits from defects, or the positions where check operators give wrong signs. These models have two parameters; the temperature T and the probability of "wrong-sign" bond (plaquette) p, and it is shown that error correction can be done successfully in magnetically ordered phase in p-T plane of RBIM (RPGM), which means the preservation of topological encoded quantum information, while in disordered phase error correction cannot be done properly and encoded information is lost. The transition point gives the accuracy threshold of error correction.

Here we consider the random 4D Z_2 RPGM, which can also be regarded as the model of topological quantum memory. For example, the model of topological quantum memory on the 3D square lattice with time valuable can be viewed as the 4D RPGM.

The 4D gauge model has a good property; It is known that the 4D nonrandom gauge model is self-dual [4]. We will analyze the 4D RPGM using self-duality. The formalism of duality transformation here is based on the one by Wu and Wang [5]. They showed the self-duality of partition function using Fourier transformation for 2D nonrandom multi-component spin model. Recently it is shown that the Wu-Wang duality is applicable to studying the 2D random spin model in Ref.[6]. Here we show that analysis using self-duality is also useful for the 4D RPGM. We conclude that the accuracy threshold of the 4D RPGM is the same as that of the 2D RBIM.

We consider the 4D Z_2 gauge model. The Hamiltonian is written as,

$$H = -J\sum_P U_P, \qquad U_P = \sum_{l\in\partial P}\xi_l, \tag{1}$$

where P and l are plaquettes and bonds in the 4D square lattice respectively, and J is the uniform coupling constant. ξ_l is a modulo 2 random valuable which takes 0 or 1. The partition function is,

$$Z = \sum_{\xi_l=0,1}\prod_P u(U_P), \qquad u(U_P) = \exp[KU_P(\xi_l)], \tag{2}$$

where $K = \beta J = J/kT$. For taking the dual of this gauge model, we rewrite the partition function Z once again using the plaquette variables ξ_P,

$$Z(u) = \text{const.}\times \sum_{\xi_P=0,1}{}' \prod_P u(\xi_P), \quad u(\xi_P) = \exp[K\xi_P], \tag{3}$$

where the sum with prime symbol is the one with the following constraint,

$$\sum_{P\in\partial C}\xi_P = 0 \pmod 2 \qquad \text{for all } C, \tag{4}$$

where C means a 3D cube in the 4D square lattice. With this constraint, we can keep the equivalence of the two Z's in Eqs.(2) and (3).

Next we study the quantities in the dual space using self-duality of plaquette; the dual of a plaquette P (2D object) in the 4D square lattice is also a plaquette P'. We define the variables $\eta_{P'}$ on dual plaquettes, which also take the value 0 or 1, and perform Fourier transformation of $u(\xi_P)$ defined by,

$$\begin{aligned} u^*(\eta_{P'}) &= \frac{1}{\sqrt{2}}\sum_{\xi_P=0,1}\exp(\pi i\xi_P\eta_{P'})u(\xi_P) \\ &= \frac{1}{\sqrt{2}}\sum_{\xi_P=0,1}\exp(\pi i(\sum_{l\in\partial P}\xi_l)\eta_{P'})u(\xi_P), \end{aligned} \tag{5}$$

or conversely,

$$u(\xi_l) = \frac{1}{\sqrt{2}}\sum_{\eta_{P'}=0,1}\exp(\pi i(\sum_{l\in\partial P}\xi_l)\eta_{P'})u^*(\eta_{P'}). \tag{6}$$

We put Eq.(6) into Eq.(2) and take the sums over ξ_l. Considering that one bond is always contained within boundaries of six faces, exponential factors in Fourier transformation lead (modulo 2) delta functions such as,

$$\delta_{mod2}(\sum_{P'\in\partial C'}\eta_{P'}) \qquad \text{for all } C', \tag{7}$$

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where C' is the dual cube in the 4D dual square lattice. This condition is equivalent to Eq.(4) and we can replace the summation over $\eta_{P'}$ with the constrained one.

Now we obtain the partition function using dual valuables,

$$Z(u^*) = \text{const.} \times \sum_{\eta_P=0,1}{}' \prod_{P'} u^*(\eta_{P'}). \tag{8}$$

Two partition functions in Eqs.(3) and (8) are related as $Z(u) = \text{const.} \times Z(u^*)$. From this relation, we can get the transition point $\beta_c = \frac{1}{2}\ln(1+\sqrt{2})$ easily.

Next we explain the technique to deal with RPGM, which is proposed in Ref.[6]. They studied the 2D ($\pm J$) RBIM using Wu-Wang duality, and our method for analyzing the 4D RPGM is essentially the same as theirs.

We incorporate randomness in gauge model Eq.(1),

$$H = -J\sum_P \tau_P U_P, \tag{9}$$

where τ_P is plaquette-dependent random variable. Here we assume that τ_P takes 1 with probability p and -1 with $1-p$.

We make use of the replica technique for averaging random τ_P, that is, we prepare n replica systems and take the $n \to 0$ limit to obtain physical quantities. After taking the configuration average over random valuables, the partition function is determined by local plaquette Boltzmann factors because of homogeneity of system.

Here we define the factor x_k, which corresponds to the averaged plaquette Boltzmann factor of arbitrary plaquette for the configuration $\prod_l U_l = 1$ in $n-k$ replicas and -1 in k replicas. The explicit form of x_k is,

$$x_k = p{\kappa_+}^{n-k}{\kappa_-}^{k} + (1-p){\kappa_+}^{k}{\kappa_-}^{n-k}, \tag{10}$$

where $\kappa_\pm = e^{\pm K}$. The averaged partition function Z_n depends on averaged plaquette Boltzmann factors x_k.

In the dual lattice, we can also define the dual averaged plaquette Boltzmann factor ${x_k}^*$ for n replicas similarly,

$$\begin{aligned} 2^{n/2}x^*_{2m} &= (\kappa_+ + \kappa_-)^{n-2m}(\kappa_+ - \kappa_-)^{2m}, \\ 2^{n/2}x^*_{2m+1} &= (2p-1)(\kappa_+ + \kappa_-)^{n-2m-1}(\kappa_+ - \kappa_-)^{2m+1}. \end{aligned} \tag{11}$$

We can write the duality of partition function using x_k and x_k^*,

$$Z_n(x_0, x_1, \dots x_n) = Z_n(x_0^*, x_1^*, \dots x_n^*), \tag{12}$$

where we neglect the trivial factor, which is irrelevant to thermodynamic property.

This procedure is completely the same as used for the 2D RBIM. In this model multicritical point lies on Nishimori line defined by $e^{-2K} = (1-p)/p$ and also on the transition line between the paramagnetic (disordered) phase and the ferromagnetic (ordered) phase. In Ref.[6], they conjectured that the multicritical point lies on the line defined by averaged Boltzmann factor,

$$x_0 = x_0^*, \tag{13}$$

and they checked this relation and Nishimori condition lead exact multicritical point for small n. Similarly, we expect Eq.(13) and Nishimori condition give the probability p_c at the multicritical point for $n \to 0$. p_c is estimated at $0.88997\dots$ from these equations, which is in good agreement with the numerical studies in Ref.[7].

This argument for the 2D RBIM is also applicable to the 4D RPGM. In this model order parameter is defined by $W[C] = \prod_{l\in C} U_l$ where C is arbitrary closed loop. Random and thermal averaged $W[C]$ obeys the perimeter law for ordered (Higgs) phase or area law for disordered (confinement) phase. Higgs and confinement phases correspond to ferromagnetic and paramagnetic phase respectively. If we apply the previous discussion for the 2D RBIM to the 4D RPGM, we automatically obtain the phase diagram for the 4D RPGM from the one for the 2D RBIM by replacing the names of phases.

In conclusion, we discussed the accuracy threshold of error correction using the 4D RPGM, which is equivalent to the toric code in 4D (or in 3D with repeated observations). The boundary of ordered and disordered phases is the same as that of the 2D RBIM, and we can apply the knowledge of transition property of the 2D RBIM also to the 4D RPGM. We can also expect that this phase boundary gives the limitation to the boundary of ordered phase for the 3D RPGM, because the reduction of dimension always enhances disorder. The transition point at $T = 0$ is given in Ref.[3] by numerical study, which also supports our expectation.

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