Quantum coherence in the presence of unobservable quantities

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Abstract. State representations summarize our knowledge about a system. When unobservable quantities are introduced the state representation is typically no longer unique. However, this non-uniqueness does not affect subsequent inferences based on any observable data. We demonstrate that the inference-free subspace may be extracted whenever the quantity's unobservability is guaranteed by a global conservation law. This result can generalize even without such a guarantee. In particular, we examine the coherent-state representation of a laser where the absolute phase of the electromagnetic field is believed to be unobservable. We show that experimental coherent states may be separated from the inference-free subspaces induced by this unobservable phase. These physical states may then be approximated by coherent states in a relative-phase Hilbert space.

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1 Introduction

The representation of a state and its associated interpretation are fundamental issues in quantum mechanics. The state representation of a system summarizes our knowledge about that system; it summarizes the information about any observable we wish to measure. Conversely, experiments on a system allow us to determine the associated state representation.

Now it turns out that some observables are not accessible to experiment. Such quantities might be called *unobservables*, a term which we shall take to be a synonym for unmeasurable quantities. There are several mechanisms by which a quantity may be unobservable. One rigorous mechanism involves global conservation laws. In particular, the Wigner-Araki-Yanase (WAY) theorem says that any operator which does not commute with an operator corresponding to a global conservation law is not observable [1]. Therefore any system satisfying global conservation laws involves unobservable quantities.

In addition to those quantities whose unobservability is directly associated with global conservation laws, there are other quantities for which no rigorous argument currently exists guaranteeing their unobservability, but which are, nonetheless, generally accepted to be inaccessible to any experimental means. An example of such a quantity is the absolute phase of the electromagnetic field [2].

What are the consequences for the state representation of a system when unobservable quantities are involved? In this paper we investigate the consequences of unobservable quantities on the state representation more fully. To do this, we consider the possibility that the non-uniqueness can be isolated in the state representation. More particularly, that the state may be written in a form of a direct product of two states or components, where at least one of these components involves no non-uniqueness due to unobservability. This component involves a subspace which is spanned by only eigenstates of truly observable quantities. By contrast, the minimal non-unique component corresponds to that part of the state representation which cannot be determined by any experimental procedure; it corresponds to an inference-free subspace.

We start by considering the case where the unobservability of quantities is guaranteed by global conservation laws and demonstrate that the inference-free subspace may be isolated and separated from the remainder of the state representation. Then we proceed to a generalization of this case. In particular, we look into the case of the laser output field, where its absolute phase is generally accepted to be unobservable, though a direct proof for this based on the WAY theorem is currently lacking. Notwithstanding this, we demonstrate how the inference-free component of the state description may be isolated as an excellent approximation.

2 the Wigner-Araki-Yanase theorem and an inference-free subspace

The Wigner-Araki-Yanase (WAY) theorem gives us a playground for systems where unobservability of certain quantities is guaranteed by global conservation laws. The WAY theorem states that any operator which does not commute with an operator of the global conservation is not observable [1]. However, the WAY theorem also suggests that a relative quantity of the unobservable absolute operators can be observable, as such a relative quantity would commute with the conserved quantity. By constructing a subspace where the relative quantity is well-defined, we isolate the inference-free component. Here we show an example which can be easily generalized. A relative position operator of the observed system to the apparatus $\hat{x}_r$ commutes with the total momentum $\hat{\Pi}$ and hence is observable. We take eigenstates of an operator $\hat{x}_a$, the sum of the absolute positions of the observed system and the appartus, to construct the entire Hilbert space together with eigenstates of $\hat{x}_r$. The Hilbert space for the entire system (the observed system and the apparatus) can be expanded by $\{|x_r\rangle \otimes |x_a\rangle\}$. When a state is separable in terms of relative- and absolute- quantity subspaces, it is written in the form $|\psi\rangle = |\psi_r\rangle \otimes |\psi_a\rangle$

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where $|\psi_{r\ (a)}\rangle$ is on the Hilbert space spanned by $|x_{r\ (a)}\rangle$ respectively.

As the operator $\hat{x}_a$ is not observable, the state $|x_a\rangle$ must be considered as a label of the equivalence class of states [3]. Hence the state $|x_a\rangle\langle x_a|$ stands for a set of the states

$$\rho_a = \int dX P(X) e^{-iX\hat{\Pi}} |x_a\rangle\langle x_a| e^{iX\hat{\Pi}}, \tag{1}$$

with all possible prior distributions $P(X)$. However, the state $|\psi_r\rangle$ is preserved under the action of the displacement operator $e^{-iX\Pi}$, which allows the total density matrix to be

$$\rho = |\psi_r\rangle\langle\psi_r| \otimes \rho_a. \tag{2}$$

The relative-position state $|\psi_r\rangle$ is on the relative-position Hilbert space and the relative-quantity operators can be defined on this subspace. The state ρ_a forms an inference-free component and the inference-free subspace is constructed to be completely free from a choice of the prior distribution.

Now, let us generalize the argument to entangled states. A general state can be represented by

$$|\phi\rangle = \int dx_r dx_a \psi(x_r, x_a) |x_r, x_a\rangle. \tag{3}$$

In the case, by contrast to the separable case, it seems non-trivial to construct an inference-free subspace. To generate such entangled states from a separable state, it is necessary for the system to start with a superposition of the total momentum eigenstates. The state preparation process for such a superposition must be consistent with the total momentum conservation. However it is easy to see that none of the operators which generate a superposition from an eigenstate is allowed under the conservation law. Any creation of superposition necessitates a third system to be involved in the state preparation process. Thus considering the state preparation process, only the eigenstates of the total momentum are consistent with the global symmetry and hence the state is always separable and the relative-position subspace always can be constructed. This constraint on states allowed in the system could be considered as a superselection rule.

3 The case of laser output field

In this section, we generalize the argument to cases where unobservability is not guaranteed by global conservation laws. Our particular interest of such cases here is the laser output field. Despite a lack of rigorous proof, the absolute phase of an electromagnetic field has been considered to be non-observable [2, 4]. Due to the non-observability of absolute phase ϕ, the state representation is often averaged over this unobservable quantity as

$$\rho = \int_0^{2\pi} \frac{d\phi}{2\pi} P(\phi) \big|\, |\alpha| e^{-i\phi} \big\rangle \big\langle |\alpha| e^{-i\phi} \big| \tag{4}$$

where $P(\phi)$ is an untestable prior distribution function. The argument from the section 2, suggests that we should generalize this state to having two modes in order to factor out the relative phase from the two-mode coherent state.

A two-mode coherent state is given as

$$\begin{aligned} |\alpha, \beta\rangle &= \big|\, |\alpha| e^{-i\phi_\alpha} \big\rangle \otimes \big|\, |\beta| e^{-i\phi_\beta} \big\rangle \\ &= e^{-\frac{\langle \hat{N}\rangle}{2}} \sum_{N=0}^{\infty} \sum_{M=-\frac{N}{2}}^{\frac{N}{2}} \frac{\alpha^{\frac{N}{2}+M} \beta^{\frac{N}{2}-M} |N, M\rangle}{\sqrt{(\frac{N}{2}+M)!(\frac{N}{2}-M)!}} \end{aligned} \tag{5}$$

Here N is the total photon number of the state, i. e. $\langle\hat{N}\rangle = |\alpha|^2 + |\beta|^2$, and M is the difference of the photon number of each mode, which is either integer (for even total photon numbers) or half-integer (for odd total photon numbers). Obviously this state is not separable in terms of the two subspaces, $\{|N\rangle\}$ and $\{|M\rangle\}$. Thus, perfect separation between the physical and inference-free components is not possible. However, as we shall see, this may be achieve to an excellent approximation.

Take a set of parameters as $\phi_\alpha - \phi_\beta = \phi_r$, and $\xi = -\frac{|\alpha|}{|\beta|} e^{-i\phi_r}$. The two mode coherent state can be written as the sum of spin coherent states, yeilding

$$|\alpha, \beta\rangle = e^{-\frac{\langle\hat{N}\rangle}{2}} \sum_{N=0}^{\infty} \frac{\left(\langle\hat{N}\rangle^{1/2} e^{-i\phi_\beta}\right)^N}{\sqrt{N!}} |N\rangle \otimes |\xi\rangle_N, \tag{6}$$

where $|\xi\rangle$ is a spin-$N/2$ coherent state.

Here we consider a limit of large total photon number $\langle\hat{N}\rangle^{1/2} \to \infty$. When $|\alpha| \ll |\beta|$ is satisfied, the group contraction may be taken in the order of $\langle\hat{N}\rangle$. The spin size given by $|\beta|^2$ goes to infinity and then the state is contracted to a Heisenberg-Weyl coherent state $|-|\alpha| e^{-i\phi_r}\rangle$.

Thus, the coherent state from a laser may be represented approximately as

$$|\alpha, \beta\rangle\langle\alpha, \beta| \simeq |-|\alpha| e^{-i\phi_r}\rangle\langle -|\alpha| e^{-i\phi_r}| \otimes \rho_{\rm P}\ , \tag{7}$$

under the condition

$$\langle\hat{N}\rangle \simeq |\beta|^2 \gg |\alpha|^2. \tag{8}$$

Hence the state on the subspace of the relative phase is a coherent state, which is, in fact, what we call a coherent state in experiments. This state presentation for relative phase does not involve a prior distribution for the absolute phase and so circumvents any difficulty associated with its unobservability.

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