LOCC observations on quantum states

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Abstract. We investigate techniques for determining and optimizing LOCC measurements on quantum states. We give complete solutions under some symmetries, and present work in progress to extend these results to two (and maybe more) copies of Isotropic/ Werner states.

Keywords: LOCC, state discrimination, entanglement, Bell states, Werner states, Isotropic states.

In quantum information theory it is very common to envisage situations in which several parties would like to optimize measurements on joint quantum states when they are restricted to using Local Operations and Classical Communication (LOCC). Examples include 'source/target' distillation schemes [2], quantum cryptographic scenarios [1], and experimental demonstrations of entanglement [3]. However, although global quantum measurements are well characterized within the POVM formalism, there is no such understanding of measurements that can be implemented using LOCC.

This poster will broadly consist of two parts. Firstly we discuss our work in which LOCC measurements have been completely characterized for certain symmetries (including Bell diagonal states). Secondly we will discuss our attempts to extend these results to situations where we have access to two or more copies of the state.

Classes of Operations

The set of LOCC operations is notoriously difficult to determine. In order to assist in tackling this problem, Rains [5] defined the *separable* and *PPT* operations. The separable operations are those with Kraus decomposition of form $S(\rho) = \sum_n A_n \otimes B_n \rho A_n^\dagger \otimes B_n^\dagger$. The PPT operations $P(\rho)$ are those such that both P and $\Gamma \circ P \circ \Gamma$ are completely positive, where Γ denotes partial transposition of any fixed subsystem (Alice or Bob). It is possible to show the strict inclusions: LOCC $\subset$ separable $\subset$ PPT $\subset$ global ([5]). This classification is particularly elegant when it comes to measurements, as one can show that a given N-outcome POVM $\{M_k | \sum_k M_k = I, M_k \geq 0, k = 1...N\}$ can be implemented by separable operations iff the POVM elements are separable, and can be implemented by PPT operations iff the POVM elements are PPT [1, 4]. Unless explicitly stated, we will often use the terms PPT and separable interchangeably, as the results that we discuss often apply to both cases.

Using these classes to restrict Local Measurements

The implications of the previous section mean that we have one powerful way of checking whether a measurement cannot be implemented by LOCC means: a measurement cannot be implemented using LOCC if any POVM elements are inseparable. Unfortunately this condition is not sufficient, and there are some separable POVMs that cannot be implemented locally [7].

Nevertheless, it is of interest to identify how different the set of PPT POVMs is from the set of LOCC ones. We can invoke a famous result from convex analysis (the Krein-Milman theorem [8]) to state that: "A given compact convex set of POVMs is implementable by LOCC iff all its extreme points are". As both separability and PPT constraints lead to compact convex sets, we can try the following route to solving the problem:

- (1) Try to identify extremal points of the PPT POVMs under symmetry groups of interest.
- (2) Try to identify local protocols that match them.

Of course deciding whether a given POVM is separable (or even PPT) is usually a non-trivial task. However, if we know that the input states to be measured have certain symmetries, then the task can often be much easier [6]. We will rely heavily upon this fact later.

Finding the extremal points

There are two observations that are particularly useful. They are true for any convex constraint on matrices that is also closed under multiplication by non-negative scalars, in particular also for separability. The first observation is a straightforward extension of a known result for global POVMs:

> In any extremal separable POVM , the POVM elements must be linearly independent.

This result means that we can usually place a good upper bound on the number of outcomes an extremal POVM can have. Trivially it will be no greater than the dimension of the space of matrices representing the POVM. However, it cases of symmetry it can often be reduced even further, say to an integer K. Our second observation demonstrates a relationship between measurements of the full K outcomes and the 2 outcome extrema:

> Any K outcome extremal POVM must have measurement elements that are proportional to POVM elements taken from the 2-outcome extrema.

This is very useful under symmetry, where it can be easier to construct the 2-outcome extrema by other means.

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Matching the extrema by local protocols

We have few systematic techniques for performing this task. Under cases of symmetry, it often helps to write each POVM element in separable form, and then see what constraints the terms in the decomposition must satisfy. This, with a little guess work, has allowed us to match local protocols to all PPT POVM extrema from the Isotropic, Werner, Bell and Local Orthogonal symmetries [4]. In all these cases the procedures can be modified so that Alice and Bob do not need to communicate classically in order to implement the measurement, although they do need to communicate to produce the result. The detailed protocols for the Bell and Local Orthogonal cases may be found in [4]. However, we will discuss the Isotropic and Werner cases in detail, because a particularly simple form of solution is possible in the single copy case. Together with the other pleasant properties that these states possess, we are hoping that this will allow progress on measurements with multiple copies.

Local measurements on a single copy of a Werner or Isotropic state

We will only consider the Isotropic group directly here, as implications for the Werner group follow immediately from the way that partial transposition connects the two symmetries (see [6] and [4] for details). The Isotropic symmetry group is $U \otimes U^*$, where * denotes complex conjugation in a fixed local basis. Any POVM invariant under this group has elements that can be written as:

$$M_k = a_k P_+ + b_k(I - P_+) \tag{1}$$

where P_+ is the projector on the canonical maximally entangled state. In order to give PPT POVMs the $\{a_k\}_k$ and $\{b_k\}_k$ must be two probability distributions that satisfy the additional PPT requirement $(d+1)b_k \geq a_k \ \forall\ k$. It transpires that it is possible to turn any such POVM into a local measurement by a simple linear transformation. Alice and Bob need to perform the following:

Step 1 First they perform a *twirl* (see e.g [6]) over the Isotropic group $U \otimes U^*$. This can be done locally.

Step 2 Alice performs a Von Neumann measurement in the computational basis $\{|i\rangle\}$, and tells Bob the result.

Step 3 Bob then performs a POVM with elements N_k described by:

$$N_k = x_k |i\rangle\langle i| + y_k(I_B - |i\rangle\langle i|) \tag{2}$$

$$\text{with:} \quad \begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/d & (d+1)/d \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix}. \tag{3}$$

It can be shown that this measurement protocol will match any PPT POVM of isotropic form. The key ingredient is the transformation 3, which maintains positivity and completeness of Bob's measurement iff the original Isotropic POVM 1 is PPT. We can show that no *simple* [4] analogous solution exists for the Bell and Local Orthogonal PPT POVMs. The advantage of this Isotropic solution is that it avoids the task of having to find the extrema, which in itself can be an extremely difficult task. The possiblity that a similar solution might be found in the situation where we have access to many copies of the Isotropic state has stimulated our current investigations.

Ongoing attempts to extend this solution to multiple copies

As this is ongoing work, we can only report preliminary results in this abstract. For the case of two identical copies of an Isotropic or a Werner state, we can apply the techniques discussed above for finding extremal measurements. Unlike the single copy case, the qubit case gives different results to higher dimensions: the set of extremal PPT measurements on two copies of a given Isotropic/Werner state has (A) 5 extremal measurements for qubit states, and (B) 7 extremal measurements for higher dimensions. We can currently match all the qubit measurements with local protocols, and 5 out of the 7 extrema for higher dimensions. It is interesting that the results seem to depend on dimension as soon as we move to two copies. It may be the case that this automatically excludes a simple solution in the manner of the single copy case, although we still cannot be sure.

Conclusion

We have discussed techniques for characterising the set of LOCC measurements on quantum states and applied them to some common symmetry groups, obtaining complete solutions. In the Werner and Isotropic case, the solution is of a very simple form, and this leads to some hope of extending the solution to multiple copies. Our initial work has been on the two copy case, where some interesting dimension dependence occurs.

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