# Extension of Schmidt rank in infinite dimensional systems

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A complete proof of Nielsen's theorem of entanglement convertibility under local operations and classical communications (LOCC) in infinite dimensional systems is given. Based on this generalization of the theorem, a new entanglement monotone which can be applied to states with infinite Schmidt rank is introduced. This monotone, rapidity of convergence of Schmidt coefficients, is a natural extension of the Schmidt rank and useful for investigating LOCC and nondeterministic LOCC entanglement convertibility in infinite dimensional systems.

Entanglement is regarded as the key resource which allows many quantum information processing schemes to out-perform their classical counterparts. The convertibility of two different entangled states (for single copy or multi-copy cases) under LOCC (local operations and classical communications) is important for the qualitative and quantitative understanding of entanglement. For finite dimensional bipartite systems, we now have a better understanding of entanglement convertibility based on intensive work in recent years. The condition for the deterministic LOCC entanglement convertibility of two pure entangled states is given by Nielsen's theorem [1]: The necessary and sufficient condition to convert a state $|\Psi\rangle$ to another state $|\Phi\rangle$ by LOCC is that the Schmidt coefficients of $|\Psi\rangle$ are majorized by those of $|\Phi\rangle$. On the other hand, for infinite dimensional systems (or continuous variable systems), there are still many open questions on general LOCC entanglement convertibility, although there are important works [2], which have investigated a limited class of local operations (Gaussian operations) from a practical point of view.

As the first step to understanding the full potential of infinite dimensional systems for quantum information processing, we study how the results obtained for LOCC entanglement convertibility in finite dimensional systems can be generalized to infinite dimensional systems. In this paper, we deal with a general LOCC setting, which includes countably infinite measurement results. We give a complete proof of Nielsen's theorem in infinite dimensional systems and then show a new classification of entangled states, which have infinite Schmidt rank, based on our generalization of Nielsen's theorem.

To prove the sufficient part of the theorem, we first define $\epsilon$-convertibility of LOCC such that $|\Psi\rangle$ is $\epsilon$-convertible to $|\Phi\rangle$ if and only if there exists a LOCC $\Lambda$ such that $\| \Lambda(|\Psi\rangle \langle\Psi|) - |\Phi\rangle \langle\Phi| \| < \epsilon$ for any $\epsilon > 0$. Then we prove that for any $\epsilon > 0$, the sufficient condition for $\epsilon$-convertibility is that the Schmidt coefficients of $|\Psi\rangle$ are majorized by those of $|\Phi\rangle$. For necessity, we show a new simple derivation of Lo-Popescu's theorem in infinite dimensional systems. This completes the proof.

In finite dimensional systems, the Schmidt rank (the rank of the reduced density matrix) gives a necessary condition for LOCC entanglement convertibility and is an entanglement monotone. However, the Schmidt rank is not valid for classification of states with infinite Schmidt ranks in infinite dimensional systems. Since the Schmidt rank represents how fast the Schmidt coefficients disappear, it is natural to consider an extension of the Schmidt rank that represents how fast the Schmidt coefficients converge to zero. We show that this extension is a new entanglement monotone for infinite dimensional systems.

We define a real function $s^+(\Psi) > 1$ which indicates rapidity of convergence of the Schmidt coefficients as the following: For a bipartite state $|\Psi\rangle$ for which the Schmidt coefficients are given by $\{\lambda_n\}_{n=1}^{\infty}$, there exists a real number $s^+(\Psi) > 1$ such that $\overline{\lim}_{n\to\infty} n^r \lambda_n = \infty$ for $r > s^+$ and $\overline{\lim}_{n\to\infty} n^r \lambda_n < \infty$ for $r \leq s^+$. If for all $r \in N$ $\overline{\lim}_{n\to\infty} n^r \lambda_n = 0$ is valid, define $s^+ = \infty$. We also define another real function $s^-(\Psi) > 1$ in the similar way of $s^+(\Psi)$ but using $\underline{\lim}$ instead of $\overline{\lim}$. $s^+(\Psi)$ and $s^-(\Psi)$ satisfy the relation $s^+(\Psi) \leq s^-(\Psi)$.

Using our generalization of Nielsen's theorem, the necessary condition to convert $|\Psi\rangle$ to $|\Phi\rangle$ by LOCC is $s^+(|\Psi\rangle) \leq s^-(|\Phi\rangle)$. On the subspace of $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ where $s^+(|\Psi\rangle)$ and $s^-(|\Psi\rangle)$ are equal, $s = s^+ = s^-$ is a monotone function for LOCC entanglement convertibility, that is, an entanglement monotone. Further, we investigate nondeterministic LOCC [3] entanglement convertibility of states in infinite dimensional space. We show that the rapidity of convergence of Schmidt coefficients also classifies nondeterministic LOCC entanglement convertibility in infinite dimensional systems.

**Acknowledgements:** This work is supported by the Research Foundation for Opto-Science and Technology, the Sumitomo Foundation, and the Asahi Glass Foundation.

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