Numerical Test of Superadditivity of Entanglement of Formation for Four-Qubit States

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Abstract. It is well known that additivity of EoF can be derived from its superadditivity. We have numerically checked the superadditivity for four-qubit states, which is the minimum and also the only computationally viable case among nontrivial ones. No counterexample has been found by testing more than 1,000,000 randomly chosen cases.

Keywords: entanglement of formation, additivity problem, super additivity, numerical experiment

1 Entanglement, Measures, Additivity

Quantum entanglement is one of the deeply unique features of quantum mechanics and plays a key role in quantum computation and quantum information; for example, quantum teleportation [1], superdense coding [2], quantum cryptography [3] *etc.* Moreover, quantum entanglement is necessary for quantum algorithms operating to be achieved with pure states of an exponential speed-up over corresponding classical algorithms [4]. One of the basic problems of the study of entanglement itself is how to quantify entanglement and one of the most widely accepted measures of entanglement is *entanglement of formation* (EoF) [5], defined for a bipartite state ρ_{AB} in Hilbert space $H_A \otimes H_B$ as

$$E_f(\rho_{AB}) = \min \sum_i p_i S_{\text{vN}} \left(\text{Tr}_A |\Psi^i_{AB}\rangle\langle\Psi^i_{AB}| \right). \qquad (1)$$

The minimization is over all possible ensembles $[p_i, |\Psi^i_{AB}\rangle]$ satisfying some conditions like $\sum_i p_i |\Psi^i_{AB}\rangle\langle\Psi^i_{AB}| = \rho_{AB}$, to be a decomposition of ρ_{AB}. $S_{\text{vN}}(\rho) = -\text{Tr}\rho \log_2 \rho$ is the von Neumann entropy.

A long-standing open problem is whether EoF is additive for any product density matrix $\rho_{A_1B_1} \otimes \rho_{A_2B_2}$ of the Hilbert space $H_{A_1} \otimes H_{B_1} \otimes H_{A_2} \otimes H_{B_2}$, *i.e.*,

$$E_f(\rho_{A_1B_1} \otimes \rho_{A_2B_2}) = E_f(\rho_{A_1B_1}) + E_f(\rho_{A_2B_2}). \qquad (2)$$

As far as we know, no counterexample has been found so far. Moreover, this additivity has been proved for some special cases.

2 Superadditivity

Recently, another conjecture so-called (strong) *superadditivity* of EoF was formulated in Ref. [6]. Originally it was thought that it is stronger than the additivity[1], later it has been found that they are equivalent [7]. The superadditivity conjecture is stated as follows: for any four-qubit states $\rho_{A_1B_1A_2B_2}$ shared by A_1, A_2 and B_1, B_2,

$$E_f(\rho_{A_1B_1A_2B_2}) \geq E_f(\rho_{A_1B_1}) + E_f(\rho_{A_2B_2}) \qquad (3)$$

holds where $\rho_{A_1B_1} = \underset{A_2B_2}{\text{Tr}}\, \rho_{A_1B_1A_2B_2}$, and $\rho_{A_2B_2} = \underset{A_1B_1}{\text{Tr}}\, \rho_{A_1B_1A_2B_2}$. EoF itself is very hard to calculate for mixed states in most cases, and so is to check whether the superadditivity of EoF really holds, consequently.

As already pointed out in Ref. [6], if Ineq. (3) holds when $\rho_{A_1B_1A_2B_2}$ is any pure state, then it holds for any case[2](the reverse statement is trivial). Note that, at the last sentence, the Hilbert spaces $H_{A_1}, H_{B_1}, H_{A_2}$ and H_{B_2} can be fixed at your mercy.

3 Minimal Nontrivial Case: $(2 \otimes 2)^2$

For any pure state $|\psi_{AB}\rangle$, the EoF is equal to the von Neumann's entropy $\text{Tr}_B|\psi_{AB}\rangle\langle\psi_{AB}|$. Therefore it is rather easy to calculate. Moreover, also for bipartite qubits in a mixed states ρ, the EoF is easy to calculate *via* Wootters' concurrence[3] [8] $C(\rho)$, *i.e.*,

$$E_f(\rho) = H_2\left(\frac{1+\sqrt{1-C^2(\rho)}}{2}\right), \qquad (4)$$

where $H_2(x) = -x\log x - (1-x)\log(1-x)$ is the binary Shannon entropy. Therefore, Ineq. (3) is for any pure state on four qubits easily numerically verifiable.

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[1] The superadditivity presented as Ineq. (3) leads to the $\leq$ direction of Eq. (2). On the other hand, the $\geq$ direction is easy from the definition as Eq. (1). Therefore Eq. (2) follows from the superadditivity.

[2] For any mixed state $\rho_{A_1B_1A_2B_2}$, $E_f(\rho_{A_1B_1A_2B_2}) = \sum_i p_i E_f(|\Psi^i_{A_1B_1A_2B_2}\rangle\langle\Psi^i_{A_1B_1A_2B_2}|) \geq \sum_i p_i [E_f(\rho^i_{A_1B_1}) + E_f(\rho^i_{A_2B_2})] \geq E_f(\sum_i p_i \rho^i_{A_1B_1}) + E_f(\sum_i p_i \rho^i_{A_2B_2}) = E_f(\rho_{A_1B_1}) + E_f(\rho_{A_2B_2})$, where $\left[p_i, |\Psi^i_{A_1B_1A_2B_2}\rangle\right]$ is the optimal decomposition of $\rho_{A_1B_1A_2B_2}$ for EoF, and $\rho^i_{A_1B_1} = \underset{A_2B_2}{\text{Tr}} |\Psi^i_{A_1B_1A_2B_2}\rangle\langle\Psi^i_{A_1B_1A_2B_2}|$, as well as $\rho^i_{A_2B_2} = \underset{A_1B_1}{\text{Tr}} |\Psi^i_{A_1B_1A_2B_2}\rangle\langle\Psi^i_{A_1B_1A_2B_2}|$. Here, the two inequality signs are obtained from the assumption and the convexity property of EoF, respectively.

[3] The concurrence $C(\rho)$ is defined as: $C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ where the λ_i's are square roots of the eigenvalues of $\rho\tilde{\rho}$ in the descending order, and $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, where $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is one of the Pauli matrices.

4 Numerical Results

In Fig. 1, we depict the result of our test of superadditivity for 8,000 uniformly chosen random points from the whole four-qubit states.

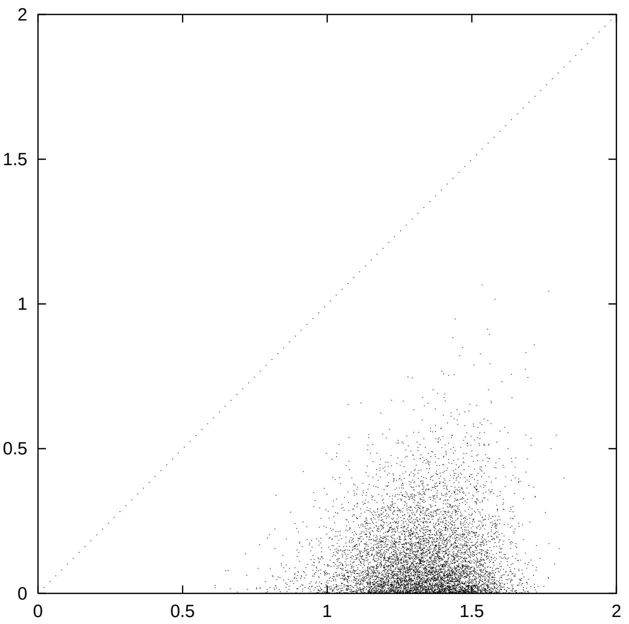

Figure 1: 8,000 states have been randomly chosen from the whole space of four qubits (16 dimensional in $\mathbb{C}$). For each of the randomly chosen states, the corresponding point is plotted at $(x, y) = (lhs., rhs.)$ of Ineq. (3). No violation of the inequality $lhs. \geq rhs.$ has been found.

We used several ways of choosing random points. In all cases as many as 1,000,000 cases, violation of the inequality has been found. Therefore, it seems very unlikely that the superadditivity could be disproved numerically.

Looking at Fig. 1, one can derive another useful fact. It concerns the question which of the values i) the difference $E_f(\rho_{A_1B_1A_2B_2}) - (E_f(\rho_{A_1B_1}) + E_f(\rho_{A_2B_2}))$ or ii) the ratio $(E_f(\rho_{A_1B_1}) + E_f(\rho_{A_2B_2}))/E_f(\rho_{A_1B_1A_2B_2})$ is more meaningful to analyze Ineq. (3). It seems the former i), because the density of the points in Fig. 1 shows a decrease toward the diagonal line according to the value i) rather than ii). Another experiment with as many as 100,000 random points more precisely agrees with this observation. Thus, we made a calculation program to seek the states which gives local minimum reference value i). The condition if any of the states which gives local minimum reference value(s) is in the tensor product form of $\rho_{A_1B_1A_2B_2} = \rho_{A_1B_1} \otimes \rho_{A_2B_2}$ is a sufficient condition of Ineq. (3), thus we are trying to analyze such states. At this moment, no violation of this tensor product condition is found from a few experiments.

5 Conclusion and Further Directions

Because of the calculation-explosion, it is very hard to numerically check the superadditivity of EoF for general multipartite qubits larger than 4-partite case. Thus, this 4-partite case is the only nontrivial one that is realistic to calculate.

Our results indicate that, even if there is a counterexamples to superadditivity among states of four qubits, such violation of the superadditivity is very rare.

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