Generalized Grover Database Search Operator with Arbitrary Initial State

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Abstract. In this paper we present a generalized description of the Grover operator, employed in a quantum database search algorithm. Our purpose is to employ an operator to find the proper solution(s) originated from an arbitrary initial state in the two dimensional vector space, where the orthogonal basis vector system are rotated in such a way where the result is obtained with a probability of error is very close to zero. Furthermore, we also stress to get the result –if it is possible– by a single try.

Keywords: Grover's Algorithm, Quantum computing, Quantum Signal Processing

1 Introduction

L. K. Grover published his fast database searching algorithm first in [1] using the diffusion matrix approach to illustrate the effect of the Grover operator, that took $\mathcal{O}(\sqrt{N})$ iterations to carry out the search, which is the optimal solution, as it was proved in [2]. Boyer, Brassard, Hoyer and Tapp [3] enhanced the original algorithm for more than one marked entry in several number of identical solutions in the database and introduced upper bounds for the required number of evaluations.

After a short debate Bennett, Bernstein, Brassard and Vazirani gave the first proof of the optimality of Grover's algorithm in [4]. The proof was refined by Zalka in [5].

Within this paper we combined and enhanced the results for generalized Grover search algorithm in terms of arbitrary initial distribution, arbitrary unitary transformation, arbitrary phase rotations and arbitrary number of marked items to build a method to construct an unstructured data base search algorithm which can be included inside a quantum computing system. Because its constructive nature this algorithm is capable to get any amplitude distribution at its input, provides sure success in case of measurement and allows to connect its output to another algorithm if no measurement is performed. Of course, this approach assumes that the initial distribution is given and it determines all the other parameters according to the construction rules.

A rather useful extension of the Grover algorithm when we decided to find minimum/maximum point of a cost function. Dürr and Hoyer suggested the first statistical method and bound to solve the problem. Later based on this result Ahuya and Kapoor improved the bounds. A further beneficial exertion possibilities of the Grover algorithm can be employed in Telecommunication field. The present authors introduced the Grover database search based multiuser detection in WCDMA environment in [6], and [7].

The rest of the paper is organized as follows. In Section 2. we introduce the generalized Grover operator ($\mathcal{Q}$) where the optimal number of iteration is determined in Section 3. The paper is closed with a final conclusion in Section 4.

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2 Generalized Grover Algorithm

Consider a large unsorted database, which contains N entries, to find the desired value with any classical algorithms would need at least $\mathcal{O}(N)$ steps.

According to the utilization of Grover's database search algorithm in practice, larger quantum systems should be taken into account where the input index register of the algorithm is given as an arbitrary output state of a former circuit and the output of the algorithm can feed another circuit without any measurement. Hence the exact knowledge of the index register after the final iteration is of great significance.

2.1 Generalization of the Original Grover's Database Search Algorithm

In [1] the Grover operator was originally defined. Henceforth let us apply some new necessary definitions and practical considerations.

1. From now onward $\mathcal{H}$ should be replaced by an arbitrary unitary transformation $\mathcal{U}$ ($\mathcal{H} \rightarrow \mathcal{U}$).

2. The original Gorver-operator ($\mathcal{O}$) is altered to
$$\mathcal{O} \rightarrow \mathcal{I}_\beta \triangleq \mathcal{I} + \left(e^{\jmath\phi} - 1\right) \sum_{x\in S} |y\rangle\langle y|. \quad (1)$$

3. Analogue to the Oracle above, the controlled phase gate ($\mathcal{P}$) is changed to
$$\mathcal{P} \rightarrow \mathcal{I}_\eta \triangleq \mathcal{I} + \left(e^{\jmath\theta} - 1\right) |\eta\rangle\langle\eta|. \quad (2)$$

4. Furthermore the initial state of index register is considered as
$$|\gamma\rangle \triangleq \sum_{x=0}^{2^n-1} \gamma_x |x\rangle, \quad (3)$$
where $\sum_{x=0}^{(2^n-1)} |\gamma_x|^2 = 1$ as appropriate.

5. Finally the two basis vectors $|\alpha\rangle$ and $|\beta\rangle$ consisting of the indexes leading to unmarked solutions and of the indexes ending in a marked entry should be redefined, that were shown e.g. in [6].

$$|\alpha\rangle = \frac{1}{\sqrt{\sum_{x\in\overline{S}} |\gamma_x|^2}} \sum_{x\in\overline{S}} \gamma_x |x\rangle, \tag{4}$$

$$|\beta\rangle = \frac{1}{\sqrt{\sum_{x\in S} |\gamma_x|^2}} \sum_{x\in S} \gamma_x |x\rangle. \tag{5}$$

Regarding the definitions in (1) and (2) the generalized Grover operation ($\mathcal{G} \to \mathcal{Q}$) looks like as follows

$$\mathcal{Q} \quad = \quad -\left(\mathcal{I} + \left(e^{\jmath\theta} - 1\right)|\mu\rangle\langle\mu|\right)\mathcal{I}_\beta, \tag{6}$$

where

$$|\mu\rangle \triangleq \mathcal{U}|\eta\rangle \tag{7}$$

and relation $\mathcal{U}^\dagger = \mathcal{U}^{-1}$ is exploited in consequence of the unitary operation property, respectively.

Before continuing our examinations, let us prove the completeness of the search.

Lemma 2.1 *If the state vectors $|\alpha\rangle$ and $|\beta\rangle$ are defined according to (4) and (5), as well as the unitary operator $\mathcal{U}$ and an arbitrary state $|\eta\rangle$ are taken in such a way that $\mathcal{U}|\eta\rangle$ lies within the vector space spanned by the state vectors $|\alpha\rangle$ and $|\beta\rangle$, then the generalized Grover operator ($\mathcal{Q}$) preserves this 2-dimensional vector space.*
For any $|v\rangle \in V$, $\mathcal{Q}|v\rangle \in V$ is true.

3 Required Number of Iterations in the Generalized Grover's Search Algorithm

After being acquainted with the two dimensional generalized Grover operator $\mathcal{Q}$, the optimal number of iterations l_{opt} during a search should be derived. Since $\mathcal{Q}$ is an unitary operator and therefore it is a normal operator, hence it has a spectral decomposition

$$\mathcal{Q} = q_1|\psi_1\rangle + q_2|\psi_2\rangle, \tag{8}$$

where $q_{1,2} = -e^{\jmath\left(\frac{\theta+\phi}{2}\pm\Delta\right)}$ denote the eigenvalues of $\mathcal{Q}$ and $|\psi_{1,2}\rangle$ stand for the eigenvectors of $\mathcal{Q}$, respectively.

Due to the spectral decomposition and the relation $\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_1\rangle = 0$,

$$\mathcal{Q}^l \quad = \quad q_1^l|\psi_1\rangle\langle\psi_1| + q_2^l|\psi_2\rangle\langle\psi_2|, \tag{9}$$

where l can be derived by

$$\langle\alpha|\mathcal{Q}^l|\gamma\rangle = 0, \tag{10}$$

which is fulfilled if both –the real and the imaginary– part of (10) are equal to zero. The imaginary one is equal to

$$\begin{aligned}\Im\left\{\langle\alpha|\mathcal{Q}^l|\gamma\rangle\right\} \quad &= \quad \cos\left(\Lambda_\gamma - \Lambda + \frac{\phi}{2}\right)\sin(2z)\sin(\Omega_\gamma) + \\ &+ \quad \cos(\Omega_\gamma)\cos(2z) = 0,\end{aligned} \tag{11}$$

since $\sin(l\Delta) \neq 0$.

3.1 Optimal Number of Iterations

Now, the way is open to determine l from (11) which provides a measurement with $P_\varepsilon = 0$. Considering the matching condition just as after some calculi

$$l\Delta = \pm\frac{\pi}{2} \pm i\pi - \arcsin\left(\sin\left(\frac{\phi}{2} - \Lambda + \Lambda_\gamma\right)\sin(\Omega_\gamma)\right). \tag{12}$$

Unlike the basic algorithm where $i > 0$ could result in a more accurate measurement in case of the generalized algorithm $i = 0$ provides $P_\varepsilon = 0$.

The number of iteration l_{MC} can be expressed from (12) as

$$l_{MC} = \frac{\frac{\pi}{2} - \left|\arcsin\left(\sin\left(\frac{\phi}{2} - \Lambda + \Lambda_\gamma\right)\sin(\Omega_\gamma)\right)\right|}{\Delta}, \tag{13}$$

which leads to a measurement with a proper solution with as high accuracy as possible. In addition we claim the following restriction on the angle Δ

$$\begin{aligned}\cos\Delta \quad &= \quad \cos\left(\frac{\theta-\phi}{2}\right) + \\ &+ \quad \sin^2(\Omega)\left(\cos\left(\frac{\theta+\phi}{2}\right) - \cos\left(\frac{\theta-\phi}{2}\right)\right).\end{aligned} \tag{14}$$

4 Conclusions

In this paper we introduced a new generalized Grover operator description applied in quantum database search algorithm. We reviewed the basic Grover database search algorithm and have shown a generalized Grover Operator where the proper solution can be determined starting from an arbitrary initial state in the vector space $\mathcal{V}$ with. We also have pointed out that under certain conditions it is possible to perform a quantum database search using only a single iteration with possibility of success equal one after a measurement.

References

[1] L.K. Grover, "A fast quantum mechanical algorithm for database search," *Proceedings, 28th Annual ACM Symposium on the Theory of Computing*, pp. 212–219, May 1996. quant-ph/9605043.

[2] C. Zalka, "Grover's quantum searching algorithm is optimal," quant-ph/9711070v2, December 1999.

[3] M. Boyer, G. Brassard, P. Hoyer, A. Tapp, "Tight bounds on quantum searching," *Proceedings 4th Workshop on Physics and Computation*, vol. 46, no. 4-5, pp. 36–43, 1996. Also in Fortschritte der Physik, Vol. 46, No. 4-5, 1998, pp. 493-505 quant-ph/9605034.

[4] C. H. Bennett, E. Bernstein, G. Brassard, U. Vazirani, "Strengths and weakness of quantum computing," *SIAM Journal on Computing*, vol. 26, no. 5, pp. 1510–1523, 1997. e-print quant-ph/9701001.

[5] C. Zalka, "Simulating quantum systems on a quantum computer," *Phys. Rev. A.*, pp. 313–322, 1998.

[6] S. Imre, F. Balázs, "Non-coherent multi-user detection based on quantum search," *IEEE International Conference on Communication (ICC)*, 2002.

[7] S. Imre, F. Balázs, "A tight bound for probability of error for quantum counting based multiuser detection," *IEEE International Symposium on Information Theory (ISIT'02)*, pp. 43, Juni 30- July 5 2002.